**Opportunities to create proofs – Dr. Paul Brown**

Let us view activities in our mathematics classrooms as opportunities to create proofs. After all, proof is unique to mathematics, and creativity is the greatest fun.

For example, as soon as our students know how to expand expressions in brackets, challenge them to prove that the product of two odd numbers is odd. We can scaffold this by showing that the product of two even numbers is even, along these lines:

For positive integer *n*, every even number can be expressed as 2*n*. The product of two even numbers 2*n*1 and 2*n*2 is 4*n*1*n*2 which can be expressed as 2×(2 *n*1*n*2). As the product has 2 as a factor, it is even.

Now, leave to your students the satisfaction of proving that the product of two odd numbers is odd.

Here are three other challenges that your students may enjoy proving, each given with a suggested solution that you should cover over while you first verify the statement with a few examples, and then attempt the proof yourself!

For any two numbers, the sum of their squares is never less than twice their product.

(*a* – *b*)2 ≥ 0

*a*2 – 2*ab* + *b*2 ≥ 0

*a*2 + *b*2 ≥ 2*ab*

The sum of any positive number and its reciprocal is at least 2.

Any square ≥ 0

(*x* – 1)2 ≥ 0

*x*2 – 2*x* + 1 ≥ 0

*x*2 + 1 ≥ 2*x*

 ≥ 2 (dividing by *x*, where *x* is a positive number)

 ≥ 2 provided *x* is positive

Construction of this proof is just the reverse of simplifying the statement that was required to be proved. The proof can also be accomplished using diagrams. The excellent “Proofs Without Words” (Nelsen, 1997, p.62) does so in four quite different ways.

The average of the squares of *a* and of *b* is never less than the square of the average of *a* and *b*.

*a* *b*

2*a*2 + 2*b*2 ≥ (*a* + *b*)2







We should not restrict ourselves to geometric proofs, nor be distracted by interesting but destructive “false proofs” unless there is a teaching point to be made from them. Instead, we should examine any certainties we insist the students memorise and ensure they are proved, however informally.

For example, we teach the “rule” that a number is divisible by three if the sum of its digits is divisible by three. When we make this claim we should let the students experiment with a few cases, but then we should prove it:

Let N be the number made up of digits a0 in the “units” column, a1 in the “tens” column, a2 in the “hundreds” column, and so on.

The value of N is an10n + an-110n-1 + ... + a3103 + a2102 + a110 + a0

The numbers 1, 10, 100, 1000, 10000 ... have a special feature which helps us prove that the rule for divisibility by 3 always is true. The feature is:

1 = 30+1 10 = 33+1 100 = 333+1 1000 = 3333+1 and so on, forever. The powers of 10 always are equal to 3 something + 1. We will let the “something” number be called k0, k1, k2, k3 and so on, all the way up to kn. We now have a new way to write the value of N.

N = an(3kn+1) + an-1(3kn-1+1) + ... + a3(3k3+1) + a2(3k2 +1) + a1(3k1+1) + a0(3k0+1)

Let’s expand the brackets. Because this is algebra we are allowed to leave out the signs. It makes this easier to type!

N = 3knan + an + 3kn-1an-1 + an-1 + ... + 3k3a3 + a3 + 3k2a2 + a2 + 3k1a1 + a1 + 3k0a0 + a0

Now we group together all the terms that have a 3 in them.

N = (3knan + 3kn-1an-1 + ... + 3k3a3 + 3k2a2 + 3k1a1 + 3k0a0) + (an + an-1 + ... + a3 + a2 + a1 + a0)

We can take out the common factor 3 from the terms in the left bracket.

N = 3(knan + kn-1an-1 + ... + k3a3 + k2a2 + k1a1 + k0a0) + (an + an-1 + ... + a3 + a2 + a1 + a0)

Obviously the left bracket is divisible by 3, so N will be divisible by 3 if the right bracket is divisible by 3. But the right bracket is simply a way to write “the sum of all the digits of N”.

Now that you can prove the rule for divisibility by three, you might like to move on to the rule for divisibility by nine.

Once you have seen it, a proof does not look hard to do. With the right help, students enjoy constructing proofs, and modifying them to communicate well and to be elegant. For teachers, the most difficult task may be finding and presenting the challenge in the first place. The posters, games and activities in my “Squares” material have been successful in initiating a lot of demand for proof. For example, Difference Squares leaves an unanswered question:

“Find the difference between the numbers on each side of the square, and write

these differences halfway along the side. Inside the original square, draw a new

square that has these differences as its corners.”

A step-by-step example is given, starting with the numbers 2, 5, 23 and 10 at the corners, and concluding with this diagram:



Students are then challenged to test Difference Squares of their own, asked to record their findings, to test whether the findings extend to hexagons and octagons, and challenged to prove their findings.

For Difference Squares, the findings that students observe might include these ideas:

A. When there are two zeros (as in the diagram) the reduction to four zeros will take four iterations at most.

B. The use of vast initial numbers does not guarantee a large number of iterations.

C. The initial numbers are equivalent to three numbers and a zero.

D. Difference Squares always reduce to four zeros in a finite number of iterations.

It may be useful, after a while, to provide a convention for labelling the initial values. For example, the diagram shown could be represented as [2, 5, 23, 10] [3, 18, 13, 8] [15, 5, 5, 5] [ 10, 0, 0, 10]. Also, use of a spreadsheet may hasten some of the experimentation with Idea B, but be sure to explain how to apply the absolute value function. Ideas A and C should motivate students to use algebra.

Attempting a proof of Idea D leads to quite generalised thinking about the situation. “The difference between any two adjacent non-zero values is less than the greatest of them, therefore each iteration causes all values to be smaller, and therefore leads to all the values eventually being zero.”

The “Squares and Ladders” game, also in Squares, takes place on a numbered board where those numbers that can be expressed as sums of different squares are identified. For example, 45 is given as 32 + 62 and also as 22 + 42 + 52. While students are playing the game you can obscure the addends and challenge the students to name them. Many questions can be posed: What pattern is there to the numbers which are not sums of two different squares? What pattern is there to the numbers which cannot be expressed as the sum of any number of different squares? What is the highest number that cannot be expressed as the sum of different squares? (Just for the record, the answers to these are: 4*n*+1, 2*n*, and 128.)

One idea discussed in the Teachers Notes is that any number that can be expressed as the sum of two squares in two different ways must be composite. For example, 22 + 92 is equal to 62 + 72 and is equal to the composite number 85. Another example: 22 + 112 is equal to 52 + 102 and is equal to the composite number 125.

The Teachers Notes offer that “The proof can be given in a single line: (*ac* + *bd*)2 + (*ad* – *bc*)2 = (*ac* – *bd*)2 + (*ad* + *bc*)2 = (*a*2 + *b*2)(*c*2 + *d*2) Here, the first two expressions show how the number *a*2*c*2 + *a*2*d*2 + *b*2*c*2 + *b*2*d*2 can be generated by adding two squares together in two different ways, the final expression shows that it can also be generated by obtaining the product of two factors (Beiler, 1963, p. 143).” However, Professor Kaye Stacey was quick to pick up that this is not an adequate proof, (it is almost the proof of the converse: that the product of two numbers that are the sum of two squares is itself a sum of two squares in two different ways), and the search was on to find a proper proof.

Required to prove:

For integers if then is composite.

Let where and are integers.

…… Equation 1

If both and are even, or both are odd, the sum of their squares is an even number and therefore composite. The same is true for and . Assume and are even, and and are odd. This means and are integers.

At this point, Kaye had a “Eureka!” insight: It is necessary to let be the highest common factor of and .

Let

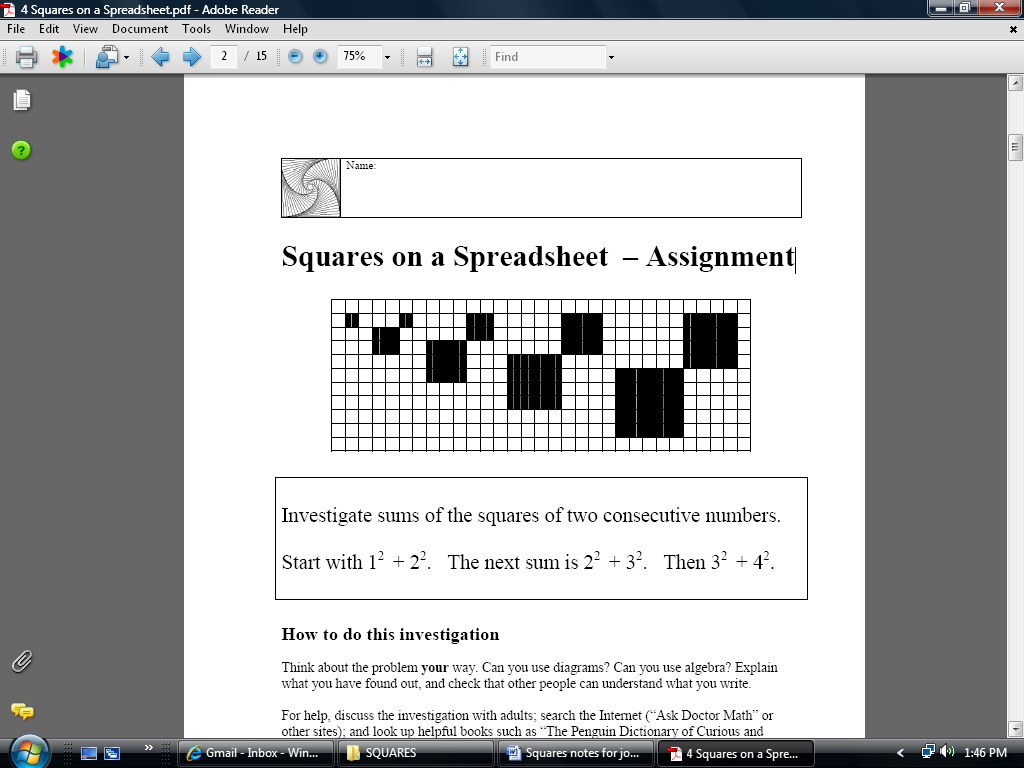
(from Equation 1)

and

The only thing that remains to be proved is that that and are integers. If all of them are integers, then the expressions in each bracket are integers, and has been shown to be the product of two integers.

Obviously, is an integer because it is the common factor of two integers and . and are found from the integers and divided by their common factor and therefore and are integers.

is also an integer. Because the right-hand side of Equation 1 is an integer, the fraction on the left-hand side must cancel down completely. That is, those factors of which are not factors of are factors of . And any factor of both and is a factor of . Therefore cancels down to be an integer, thus completing the proof. If you enjoy the formalism, you might like to add “Q. E. D.” at the end, and for once no student can claim it stands for “Quite Easily Done”.



Another Squares activity leads very naturally into a proof by induction. “Squares on a Spreadsheet”, is an investigation that produces the surprising pattern that the sums of squares of consecutive numbers often end in 13.

Required to prove: that *n*2 + (*n* + 1)2 ends in 13 for every fifth *n*.

By inspection, the conjecture is true in the case *n* = 2

Let *k* = 2 + 5*m* where *m* ∈ N

Assume the conjecture is true for *n* = *k.* That is, *k*2 + (*k* + 1)2 ends with 13.

Induction is satisfied if we can prove that (*k* + 5)2 + (*k* + 5 + 1)2 ends with 13.

(*k* + 5)2 + (*k* + 5 + 1)2 = (2 + 5*m* + 5)2 + (2 + 5*m* + 5 + 1)2

= (5*m* + 7)2 + (5*m* + 8)2

= 25*m*2 + 70*m* + 49 + 25*m*2 + 80*m* + 64

= 50*m*2 + 150*m* + 113

= 50(*m* + 2)(*m* + 1) +13

As either *m* + 2 or *m* + 1 will be even, the 50(*m* + 2)(*m* + 1) term is a multiple of 100, therefore the final two digits of the number will be 13.

Another remarkable observation is that when the sum of the squares of two consecutive numbers ends in 13, the digits that precede the 13 are triangle numbers. For example, 172 + 182 = 613, and the 6 is a triangle number. 272 + 282 also ends in 13, and produces the triangle number which is the number of balls that fit in the triangle of a pool table.

To begin the proof, we need to establish the generator formula for the triangle numbers:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Case | 1 | 2 | 3 | 4 | *n* |
| The Os  make a triangle | OX | OOX  OXX | OOOX  OOXX  OXXX | OOOOX  OOOXX  OOXXX  OXXXX |  |
| Number of symbols in the rectangle | 1 × 2 | 2 × 3 | 3 × 4 | 4 × 5 | *n* × (*n* + 1) |
| Number of Os  in the triangle | 1 | 3 | 6 | 10 |  |

Required to prove: For every fifth *n*, *n*2 + (*n* + 1)2 is a triangle number followed by 13.

The formula that generates the triangle numbers is Tn = .

From the previous proof, every fifth case of the sum of the squares of consecutive numbers

is 50(*m* + 2)(*m* + 1) +13.

The 13 occupies the final two digits of the number, therefore the digits other than the 13 are given by 50(*m* + 2)(*m* + 1) ÷ 100 =  which is equivalent to the formula that generates the triangle numbers.

I hope that a few of the observations presented here have motivated you to construct proofs, and I hope that you will frequently extend this pleasure to your students. One good resource that may help you along the way is the free introduction to proof offered by the Mathematics Enrichment Programme at http://www.cimt.plymouth.ac.uk/projects/mepres/allgcse/unitpbk.pdf

References

Beiler, A. H. (1964). *Recreations in the theory of numbers: The queen of mathematics entertains.* (2nd ed.) New York: Dover.

Brown, P. (2006). *Squares: Interesting algebra and geometry activities for secondary mathematics students*. Adelaide: Australian Association of Mathematics Teachers.

Nelsen, R. B. (1997). *Proofs without words: Exercises in visual thinking.* Washington DC: Mathematical Association of America.