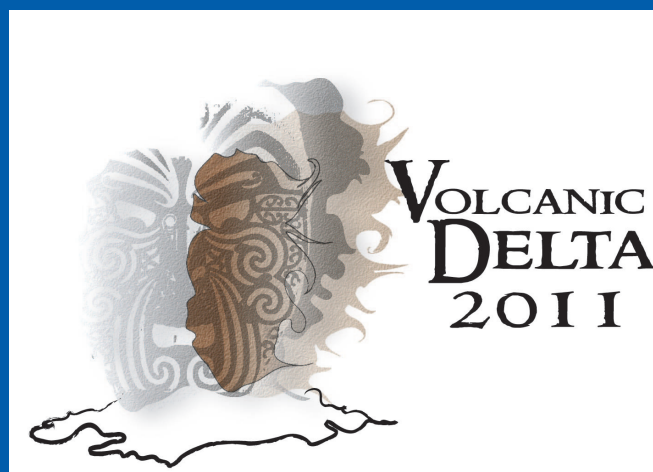


Volcanic Delta 2011
The Eighth Southern Hemisphere Conference
on Teaching and Learning Undergraduate
Mathematics and Statistics



Te Ara Mokoroa: The
Long Abiding Path of
Knowledge

Proceedings of Volcanic Delta 2011

Rotorua, New Zealand, 27 November - 2 December 2011

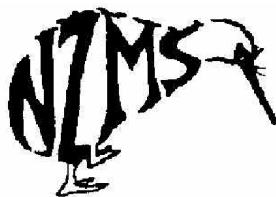
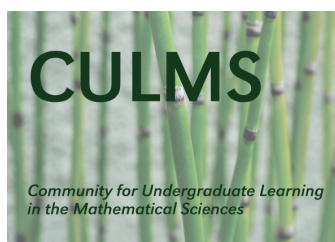
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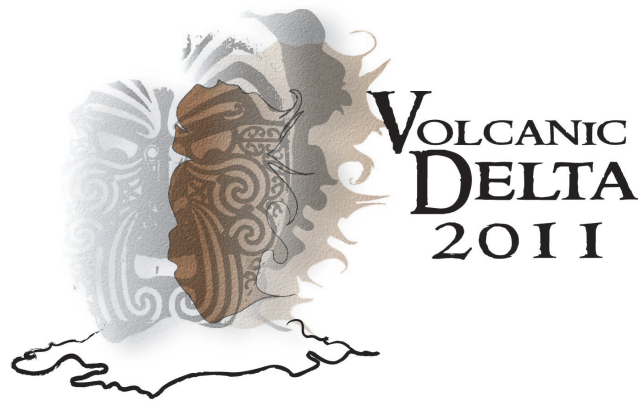
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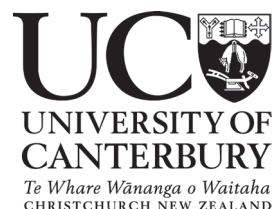


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SECTION A

iJMEST PAPERS

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SECTION B

VOLCANIC DELTA PAPERS

Challenges in Preparing Future Mathematicians for Tertiary Mathematics Teaching

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The purpose of this research project was to uncover issues and difficulties that come into play as mathematics graduate students (i.e., future mathematicians) develop their views of their roles as teachers of tertiary mathematics. Over a six-month period conversations were held with mathematics graduate students exploring their experiences and perspectives of mathematics teaching. Using hermeneutic inquiry and thematic analysis, the conversations were analysed and interpreted with attention to themes and experiences that had the potential to influence the graduate students' ideas about and approaches to teaching. Themes that are explored are: the structures of teaching assistant work, the construed nature of calculus instruction, the form of tertiary mathematics instruction, and the perceived role of professor. Lave and Wenger's notion of legitimate peripheral participation is used as framework to understand the mathematics graduate students' progression to becoming teachers of tertiary mathematics.

Keywords: teacher preparation; tertiary mathematics; graduate students

Introduction

This paper adds to the relatively new conceptualisation of mathematics graduate students (i.e., future mathematicians) as pre-service teachers of tertiary mathematics. Such a move might seem redundant – individuals who earn a PhD in mathematics will often attain an academic position in a department of mathematics and will thus be tertiary teachers of mathematics. During their doctoral programs in mathematics, future mathematicians might also have the opportunity to teach a course, giving the illusion that they are prepared for their roles as teachers. Yet there are several reasons for explicit and deliberate attention to the development of teaching practices of this group. First, researchers have pointed out the insufficient preparation of mathematics graduate students for academic careers (Bass, 2006; Chan, 2006; Shulman, 2004). Second, teaching will comprise a large portion of future mathematicians' academic work. For example, in the case of the United States, almost seventy-five per cent of mathematics PhDs will become professors at institutions dedicated to undergraduate education rather than research (Kirkman, Maxwell, and Rose, 2006), and so the development of teaching practices during graduate programs is essential in preparing mathematics graduate students for their possible future appointments.

Third, the inattention to the development of teaching practices for tertiary mathematics has a significant effect on undergraduate learners (Alsina, 2005; Kyle, 1997; National Science Foundation, 1996; Mason, 2001). In particular, Seymour and Hewitt (1997) discovered that undergraduates frequently leave mathematics and the sciences because of poor pedagogy in those disciplines, finding that “no set of problems in science, mathematics, and engineering majors was more in need of urgent, radical improvement than faculty pedagogy. All related matters, including curriculum revision, were deemed secondary to this need” (p. 165). Further, when looking at the decline of interest among science majors, the largest drop was in mathematics and teaching was cited as the cause of much of that drop (Seymour & Hewitt, 1997). Lastly, researchers have found that school teachers “are likely to teach both what and as they have been taught” (Shulman, 2004, p. 406) and despite taking teaching methods courses while attending university, “in-service teachers overwhelmingly tend to

mimic the teaching style of the mathematics courses they themselves took” (Selden & Selden, 1993, p. 440). Consequently, tertiary mathematics instruction has an impact on the form of mathematics instruction in elementary and secondary schools and has the potential to negate the impact that mathematics reforms can have in schools.

Despite the above findings, the format of tertiary mathematics teaching has remained problematic for undergraduate learners as researchers have found that most mathematicians have not diverged from the lecturing form of instruction that has caused students to leave mathematics and science (Pemberton et al., 2004). To counter this trend, recent studies have looked to prepare mathematics graduate students for their work teaching tertiary mathematics. Speer (2001) worked with mathematics graduate students in the context of reform-based undergraduate calculus. In this project, mathematics teaching assistants were expected to “teach in ways that differed substantially from their classroom experiences” (p. 1), but their beliefs about mathematics and undergraduate students counteracted the new teaching methods they learned in the reform-oriented program. DeFranco and McGivney-Burelle (2001) developed a pedagogy course where mathematics graduate students had the opportunity to address “issues surrounding pedagogy, epistemology, curriculum, and assessment” (p. 681) and were asked to make changes to their teaching practices based on what they studied in the course. However, the researchers found that the graduate students’ instruction remained “largely teacher-directed and involved very little, if any, student-student or student-teacher interaction” (p. 686). Belnap (2005) studied mathematics graduate students enrolled in a year-long, discipline-specific training course, exploring whether the course had an impact on graduate students’ views of tertiary mathematics teaching. It was found that, while the course might have some influence on the graduate students’ practices, there were other factors that could “constrain and even counteract the impact of the training program” (p. 11).

The study described in this paper also attends to the preparation of mathematics graduate students for their prospective work as teachers of tertiary mathematics, but with a different perspective. With the former conclusions in mind, the main purpose of this study was to investigate the contexts in which mathematics graduate students develop their teaching practices, and to explore their lives and experiences as they are in the process of becoming mathematicians and, most likely, teachers of tertiary mathematics. A second purpose of this study was to determine whether barriers existed for mathematics graduate students that might have the potential to prevent teacher preparation programs from bringing about hoped-for changes to their teaching practices. The questions that guided this study are:

1. How do graduate students come to understand their possible future roles as professors of mathematics?
2. What experiences do graduate students in mathematics interpret as having meaning for how they should be as mathematicians and teachers of tertiary mathematics?
3. What obstacles and issues exist for mathematics graduate students that might prevent them from embracing non-lecture based teaching practices?

The discussion will now turn to the theoretical framework and methods utilised for this study, followed by a presentation of results and discussion of the findings.

Theoretical Framework

Through their work studying apprenticeships, Lave and Wenger (1991) developed the notion of legitimate peripheral participation to name a central process by which novices gain knowledge and understanding about the practices of a community. This concept is described more fully as “learners inevitably participate in communities of practitioners and that the

mastery of knowledge and skill requires newcomers to move toward full participation in the sociocultural practices of the community” (Lave & Wenger, 1991, p. 29). They assert that “Communities of practice have histories and developmental cycles, and reproduce themselves in such a way that the transformation of newcomers into old-timers becomes remarkably integral to the practice” (p. 122). Further, they claimed “even in cases where a fixed doctrine is transmitted, the ability of the community of practice to reproduce itself through the training process derives not from the doctrine, but from the maintenance of certain modes of co-participation in which it is embedded” (p. 16). With such descriptions, the concept of legitimate peripheral participation offers an interesting lens through which to interpret and understand what might be happening for mathematics graduate students; in particular, the practices they must attend to within the community of mathematicians, the skills they must master, the modes of co-participation they must conform to in order to be considered legitimate in the community, and how these might have implications for their teaching practices.

Method of Enquiry

As it “holds out the promise of providing a deeper understanding of the educational process” (Gallagher, 1992, p. 24), hermeneutic enquiry was chosen as the method of enquiry for exploring the experiences that mathematics graduate students face in their programs. Hermeneutics helps to understand how we create and find meaning through experience and social engagement (Brown, 2001). Davis (2004) offered a description of hermeneutics as a mode of enquiry that asks, “What is it that we believe? How did we come to think that way?” (p. 206). Hermeneutic enquiry allowed the researcher to deeply explore how mathematics graduate students arrive at their teaching practices, to answer the question ‘how did they come to think that way?’ As such, hermeneutic enquiry into mathematics graduate students’ understandings, experiences, and ideas about teaching compelled a look at what is present in the structures of departments of mathematics that might cause these future professors of mathematics to adopt the teaching methods that persist as part of their role in maintaining the traditional teaching practices they experienced.

Context of the Study

Graduate students in mathematics from an urban, doctorate-granting university were approached to be participants in this study. Six agreed to participate. The group consisted of three master’s students and three doctoral students, ranging from a first semester master’s student through a final year doctoral student. Four were men, two were women, their ages ranged from 22 to 33 years, and there were four nationalities among them. While each of their paths to graduate study in mathematics was unique, all but one of the participants expected to work in academia once they completed their degrees. During their graduate programs in mathematics, each of the participants had been assigned to help students one-on-one with homework exercises in tutoring centres, grading homework assignments and exams, or leading one-hour tutorial sessions during which they presented mathematical topics similar to those in the affiliated lecture section of the course.

Carson (1986) proposed conversation as a mode of doing research within hermeneutic enquiry to explore one’s own and others’ interpretations and understandings of experience. Over a period of six months, the researcher first conducted two conversations with each participant. Then two group conversations took place with all of the research participants in attendance. These were followed by a final individual conversation with each participant. The conversations were semi-structured and recursive, meaning that while particular topics were addressed, there was flexibility in the direction of the conversations and the subsequent

conversations were informed by those previous. Each meeting was recorded and transcribed by the researcher, who listened for the topics and the language used by each of the research participants. Notes were made of the similarities in opinions and perspectives about various aspects of their experiences that appeared to be in common. Throughout the project, the research participants had the opportunity to review the analyses in a collaborative effort to refine, augment, and improve the reporting of their experiences. In the portions of dialogue cited below, pseudonyms have been used for the participants.

Analysis and Findings

Because of its recognition of the interpretive work inherent in qualitative data analysis, Braun and Clark's (2006) six-stage process for thematic analysis was coupled with hermeneutic enquiry. Thematic analysis is flexible and "has the potential to provide a rich and detailed, yet complex, account of data" (p. 78). Further, the stages of thematic analysis are in accord with Laverly's (2003) description of a hermeneutic project where "the multiple stages of interpretation allow patterns to emerge, the discussion of how interpretations arise from the data, and the interpretive process itself are seen as critical" (p. 23). The themes and the participants' comments within each theme were then assembled and analysed using a hermeneutic, interpretive lens to understand what facets of their lives in graduate school were taken as having meaning for their future work as mathematicians and teachers of tertiary mathematics. Four of the most significant themes are discussed – the structures of teaching assistant work, the construed nature of calculus instruction, the form of tertiary mathematics instruction, and the perceived role of professor.

The structures of teaching assistant work

Graduate students' first exposure to teaching undergraduates is most often through teaching assistantships, where they are expected to grade papers, tutor students, or teach courses. Such experiences are particularly powerful structures that format graduate students' ideas of their work as teachers. In the context of this study, the structures of the mathematics graduate students' work as teaching assistants prevented from them engaging in meaningful experiences with undergraduates. In particular, their work in the tutoring centre consisted of helping countless undergraduates and they spent hours repeating how to solve the same assignment questions. Emily noted the toll of this work on her desire to ensure that students understood the mathematics:

So often when they [undergraduate students] come to you... it's pretty demanding ... there's so many of them that it's – I try so hard not to give the answers away, but so often you're basically one step away. And it's nice to see them do a few steps on their own and to be able to see that process. I don't find the [tutoring centre] is very conducive to that because the second they're able to be independent, they move away because there's someone else in line. Right? Like you don't get to see that [process].

Robert's experience of the large number of help seekers was impacted by having to address the same problem, over and over again:

I don't like to teach the workshop in the sense that you don't really teach things. You're just kind of being a problem solver. You know, people have a particular problem on the assignment. ... And so I find that, a lot of TAs [teaching assistants] at the end of the day, because they're being asked the same question for the thirtieth or the fortieth time that week, that at the end of the day, they are just so tired of the question that they would just tell anyone who comes in and asks that question basically how to do it. And I kind of found that because it happened to me, too.

Steven spoke of the exhaustion that he felt in helping students one-on-one for many hours and John spoke of how the teaching assistant work was not helping him to learn how to teach because it was just a tutoring exercise.

These descriptions of their experiences are distressing given that the graduate students had initially been interested in connecting with others about mathematics. They were particularly eager to help students understand mathematics, a feeling that has been found to be common among graduate students (Golde and Walker, 2006). However, the structures of the graduate students' work in helping large numbers of students with the same problems for hours at a time prevented them from engaging in meaningful experiences with undergraduates. This was heard in Emily's account of how her work became "how fast can you turn them over" and she discovered that she needed to "plug and chug" through problems. This quickly diminished the graduate students' ability to provide the undergraduates deeper learning experiences with mathematics, leading to frustration and exhaustion. There was also a sense of disappointment of how things took place over time. In this regard, the graduate students were not able to observe the undergraduate students' progress and understanding of concepts, and so the act of tutoring was felt as an unrewarding experience.

Lave and Wenger (1991) wrote that "in apprenticeships, opportunities for learning are, more often than not, given structure by work practices instead of strongly asymmetrical master-apprentice relations. Under these circumstances, learners may have a space of 'benign community neglect' in which to configure their own learning relations with other apprentices" (p. 93). This was true for the participants in this study. They were not provided with any form of guidance for their tutoring or teaching work with undergraduates. They were without a forum to discuss their views and explore different ideas for teaching and they were left to find meaning in the structure of their experiences. They were limited in what they could do by the structures of the department and by the lack of mentoring from their faculty advisors. As a consequence, they learned about the possibilities for teaching through the limitations of their work structures, which constrained them to one way of working with undergraduate learners. The substantial impact of the structure of their teaching assistant work on their feelings for teaching was heard in Steven's statement: "I've seen [my optimism for teaching] just completely smothered. I can feel it now. It's just easier for me to not care about it."

The construed nature of calculus instruction

Because mathematics graduate students are most often assigned teaching assistantship work for first-year calculus courses, and because this work often represents their first experience working with undergraduate learners and specific mathematical content, it is important to include calculus as one of the structures that influence how mathematics graduate students envision tertiary teaching. Either because of their past experiences as calculus learners, their more recent experiences as teaching assistants for calculus courses, or a combination thereof, the participants had fixed ideas of how to teach calculus. As John remarked:

It's easy to keep teaching calculus like this. We've done it forever. We know exactly what we have to do. Almost everyone does it the same way. I mean by the time you have your PhD, you've probably been teaching calculus three or four times. You've taken it. You've TA'd for it. I mean, you know the problems, you know the classic examples. You almost don't even need a book. You can just walk up there and start teaching.

There was the sense in what John said that there is nothing left for the professor to understand, know, or learn about teaching calculus, and that there are no other directions to go. Teaching calculus is simple, rote, and effortless. Steven added the idea of calculus teaching as immutable by saying "Teaching calculus gives us this very rigid direction." Robert's statement about teaching calculus is even more telling. He said "there's no freedom in talking about it, you know what I mean. It's pretty standard stuff, you know." It seems that

the years of seeing calculus presented in a traditional, lecture-based approach have left an indelible imprint that there is but one way to offer calculus to students.

In the graduate students' voices, in knowing how to teach calculus through known problems and examples, in being limited to a certain direction, and being left without freedom as a teacher, we see Davis and Hersh's (1982) observation that in mathematics, "One also knew that the main thing was what you wrote down. As to spoken words, either from the class or from the teacher, they were important insofar as they helped to communicate the import of what was written." (p. 3). When graduate students have only their own understandings and perceptions of mathematics to work from as they develop as teachers, they have little to rely on beyond the mathematical content of the course, and they realize that they do not have a role to play in shaping mathematical content for learning.

The form of tertiary mathematics teaching

When asked what instruction in tertiary mathematics looked like and how they might prepare for it, the participants' responses alluded to the reproduction of others' teaching as well as the material in mathematics textbooks. The participants spoke of how they could rely solely on other professors' notes or on the mathematical material found in the textbooks. Steven described the structure of all mathematics courses as "definition, theory, example," where replicating the fixed structures of mathematics texts and courses was noted as a sufficient way of teaching mathematics. Robert stated that even for different sections of a course, with different students, he would "pretty much do the same thing" as his professors and "just copy his notes" onto the board, notes which were essentially a reproduction of the text. The reproduction of teaching practices in this study parallels what Seymour and Hewitt (1997) found: "teaching assistants had not received any instruction on how to teach effectively, were teaching in the same way that they themselves had been taught, and were, perforce, repeating the pedagogical errors of their professional mentors" (p. 160).

Even though their own future teaching practices would essentially be a replication of what they had experienced as learners, the participants all expressed frustration in the lack of change to tertiary teaching practices. John's statement exemplified this frustration:

It's almost like the people who could change things are so entrenched in the way they do things already, kind of the senior professors or the senior lecturers ... the chairs of departments. I mean it's, you can't as a chair of a department, I, I can't imagine you saying "You know, I think you guys should teach the discovery method this year." You know? "Our new policy is that we're not going to have lectures." Or, "In these five courses we're not going to have lectures and we're going to see what happens." I don't know that people are ready to take those chances.

What is interesting up against this statement was that John was the most experienced doctoral student in the group and had begun applying for academic positions. When asked whether he felt prepared for his future teaching work, he acknowledged that teaching a few undergraduate courses as a graduate student was sufficient preparation because he understood that teaching would not matter to his success in academia.

The perceived role of professor

The mathematics graduate students expressed a desire to be taught mathematics in meaningful ways by their own professors. Yet, in the descriptions of their professors' disinterest in for teaching, it became apparent that this type of engagement did not occur, as Steven said "we're just going through the motions in our classes," John remarked that "no one wants to be there," and Sarah revealed that the uninspiring instruction she experienced had caused her to lose her love for mathematics. Yet despite wanting to be taught in a way that would result in deeper mathematical understanding, the graduate students described a

difference between teachers and professors. John said “This is the first thing we need to get across is that professors and teachers are two completely different things,” while Steven remarked “I never really saw them as teachers. I never saw them as teachers. I always knew there was a line between teachers and professors.” Robert suggested the dissimilarity between a teacher and professor when he said “It is difficult to ask a professor to teach” – in essence, a recognition that professors do not teach. When pressed on this matter, the participants held that professors should “present material in a coherent way,” but they were not required to “help students understand the material.”

The incongruity in what they wanted as learners from their professors and how they described professors resonates with DeFranco and McGivney-Burrelle’s (2001) findings that, “although the TAs [mathematics teaching assistants] indicated a new understanding of how students learn mathematics, this belief seemed to be held peripherally and in conflict with their views about the role of teachers (i.e., to deliver information or as the central authority figure in class” (p. 687). Lave and Wenger (1991) also provide an interesting insight into what is happening for the mathematics graduate students. They stated that “learners inevitably participate in communities of practitioners and that the mastery of knowledge and skill requires newcomers to move toward full participation in the sociocultural practices of the community” (p. 29). The participants’ views of who they would be as professors were settled on what they observed in the masters of their departments, and, as a result, to the idea that they would not be teachers. Despite the complications this presented for their desires to be different from their professors, it was clear that the graduate students saw the master practitioners as not teaching and they would thus behave in similar ways once they assumed the role of professor. Chris’ statement suggests that this will be true for him, as well: “I always have these pictures in my head of when I teach it’s going to be different and I’m sure everyone does. And I’m sure it won’t be different,” signifying his future transformation from “newcomer into old-timer” (Lave and Wenger, 1991, p. 122).

Discussion

One hope for this project was to understand why education programs for mathematics graduate students had failed to instill hoped for changes in future tertiary teachers of mathematics (e.g., Belnap, 2005; DeFranco & McGivney-Burrelle, 2001; Speer, 2001). In this exploration of mathematics graduate students’ lives, it became clear that there is an intricate and complex interplay of experiences and perspectives that have the potential to work against programs for tertiary mathematics teaching. In mathematics graduate students’ initiations into teaching, both new experiences and previously formulated ideas influenced their views about teaching and how they viewed themselves as teachers. In particular, the structures of teaching assistantship work dampened the graduate students’ aspirations to provide meaningful instruction to undergraduates, leading to a sense of indifference about teaching that could be difficult reverse. Additionally, the experience in teaching calculus appeared to have a large influence on how the mathematics graduate students felt about teaching – that teaching was fixed to unalterable formats that they could not change. The sense that teaching was pre-set extended to other courses as well with one participant remarking that, in mathematics, “we teach the best possible scenario.” This perspective led to inflexibility in considering alternate practices not only for calculus, but also for the courses they wished to teach when they became professors. If the presentation of mathematics is perceived to be the best case scenario and thus does not need to be changed, there will be resistance to learning alternate ways of teaching mathematics. As well, the implication that teaching was not part of their future work had an influence on their need to be sufficiently prepared for it. This suggestion was not overt, nor was it explicitly stated anywhere. The participants did not report a public statement or even an acknowledgement that they had to abandon their own ideas about

teaching, that they should no longer consider teaching important. Rather, it seemed that the set-up, the structure of the department, and the progression to becoming a mathematician rendered it so. Finally, the graduate students held onto the position that professors do not teach. Such a perspective makes changes to teaching practices difficult when one's role is seen in this way – as one who presents mathematics, but does not teach mathematics. Since the graduate students did not view themselves as teachers, there would be little motivation for them to engage in alternate ways of teaching or connecting with learners of mathematics.

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Applying Mathematical Thinking: The Role of Mathematicians and Scientists in Equipping the New Generation Scientist

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The ability to make effective use of mathematical and statistical thinking and reasoning within context is an essential skill for graduating science students. The challenge for educators in higher education is to determine how best to foster the development of these skills. Many argue this challenge is becoming greater, given the increasingly diverse student body (often with weaker mathematics backgrounds) and the increasing use of modelling and data in modern science (meaning that the need to be able to apply mathematical and statistical thinking and reasoning is increasing). This paper discusses the implementation of initiatives within four institutions (University of Queensland, James Cook University, University of Maryland and Purdue University) that address these needs. In addition to describing the initiative itself, the change process is described. Therefore each initiative is examined through a framework based on: the need for the change, vision for the change, implementation of the change and evaluation of the change. In particular we explore the role of mathematicians and statisticians in these processes.

Keywords: interdisciplinary; higher education; science; mathematics; educational change

Introduction

Mathematical and statistical thinking and reasoning in science – the need for new curricular approaches

The increasing need for science graduates to achieve competency in the application of mathematical and statistical thinking and reasoning in science contexts has been documented extensively over the last 10 years. Publications representing collective views of modern scientists are perhaps the most powerful in highlighting the uniformity with which this belief is held. For example, *Vision and change in undergraduate biology education: A call to action* published by the American Association for the Advancement of Science in 2009 [1], and the *Learning and Teaching Academic Standards – Draft Science Standards Paper*, published by the Australian Learning and Teaching Council in 2010 [2] are two such documents which were produced after wide consultation with scientists. In each of these documents the need for science students to be competent quantitative thinkers is clearly articulated.

Despite the acknowledgement amongst scientists of the importance of the ability to apply mathematics and statistics in their profession there is a broad range of opinion amongst educators as to how best to foster these skills in school and university students. The challenge for tertiary educators is exacerbated by the downwards trend in general mathematical preparedness of students entering the sector; see Brown [3]. It is difficult to see how this trend can be reversed because of the need for the tertiary education sector to cater for an increasingly diverse range of student backgrounds to satisfy ambitious government targets for participation rates in higher education such as those in the Bradley Report [4]. In addition,

there is frequently an expectation amongst entering students that knowledge of mathematics and statistics is not essential in science. One cause of this is the common view that increasing the mathematical content of the school curriculum in disciplines such as biology may diminish their appeal to the student body [5], so that educators lean toward teaching science without emphasising the need for or links to mathematics. This expectation is reinforced in the eyes of students considering study of science beyond secondary school through the absence of mathematics prerequisites requirements for entry into science degrees at many tertiary education institutions in Australia [6]. The default position in many examples of science education at the tertiary level is that “mathematics-rich courses are presented by teaching staff from mathematics departments, and science-rich courses are taught by staff from the various scientific fields,” [7]. A more contextualised approach illustrating the application of mathematics is often proposed by secondary educators as a mechanism to motivate students to persevere with the study of mathematics. Students studying science have the perfect context in which to observe the application of mathematics, so illustrating the links between science and mathematics should be an achievable goal in tertiary education. A curriculum that fosters understanding of the intertwined nature of mathematics and science almost certainly requires approaches that are described as *multidisciplinary*, *interdisciplinary* or *integrated* (among others). The interpretation of these terms varies; see for example Venville et al. [8]. However for the purposes of this article we will assume that the nature of the material taught and the way it is taught requires either some form of collaboration across traditional discipline boundaries, or alternately, requires teachers with deep conceptual knowledge of more than one discipline area.

Implementing and analysing new curricular approaches in science

There is a lack of peer-reviewed literature discussing undergraduate science curricula that illustrate the links between mathematics and science learning outcomes at the level of the degree program [7]. Undoubtedly this situation will change as faculty engage and academic leadership embrace this issue. Factors that lead to success are of great interest to those embarking on this journey, and it is clear there is a high degree of uncertainty as to what governs success in curriculum reform. To illustrate this point it is noteworthy that literature aimed at facilitating reform in Science, Technology, Engineering and Mathematics (STEM) education is readily available at the micro-unit level. A key word search by Henderson et al. [9] revealed 295 journal articles published between 1995 and 2008 (inclusive) describing efforts of change agents to improve undergraduate STEM education. These authors report that despite significant funding for research that led to these publications, there is little evidence of widespread resulting impact. This suggests the need for systematic approaches to the analysis of educational change strategies in STEM at the degree program level within an educational framework, so that future efforts are more likely to provide greater positive impact.

Purpose of Study

It is clear that the lack of certainty about how best to develop the ability to apply mathematical and statistical thinking and reasoning in the context of science means this is an area in need of urgent attention. We present initiatives that have been used to develop these skills in science programs in each of four tertiary institutions. Whilst certainly not comprehensive, they are indicative of some current approaches in the sector, and will be of interest to those considering curriculum reform in this area.

More importantly, through these examples we aim to contribute to the body of knowledge on the implementation of new curricular approaches that facilitate the development of in-

context mathematical thinking in undergraduate science students. In particular we wish to gain insight into the working relationship between academics in mathematics and statistics and those in the science disciplines (such as biology) in the implementation of such initiatives. We aim to explore the research question:

How do cross discipline collaborations amongst mathematicians, statisticians and scientists contribute to the application of mathematical thinking of science students?

Through this publication we hope to foster interest in the discussion on how the relationship between mathematicians, statisticians and scientists can evolve to better meet the need of modern science graduates in their chosen career path.

Fullan's Model

In order to distil useful information regarding educational change it is advantageous to have access to a model that allows the complex process to be understood both in its component parts and in its entirety. This also allows for systematic comparison of different initiatives, thereby offering increased opportunity for understanding the scope for generalising to achieve educational change across context. In this article we will use a framework for analysis based on Fullan's 1982 publication [10] that examines large scale educational change. The model is presented below, in a linear form, although in reality the process is iterative. At each stage we use guiding questions to focus attention on aspects that facilitate comparison across initiatives.

- | | |
|-----------------------------|--|
| 1. Initiation of change. | Who prompts the change and why is it needed? |
| 2. Vision for change. | What does the change look like? |
| 3. Implementing for change. | How is the change translated into practice? |
| 4. Evaluating the change. | How effective is the change? |

While the model is comprehensive in its form, not all initiatives presented in this article are at the same point in their development. Hence, the analysis will focus on those areas where sufficient information exists to enable useful conclusions to be drawn. It is also important to note that the initiatives vary in target audience and intended purpose. Although this makes comparison difficult, it is still possible to comment on trends observed in the analysis.

The Institutional Initiatives

Background of the institutions

The four institutions forming the study are listed in Table 1 and are composed of public multi-campus universities in Australia and the USA. They all have a research focus; three fairly broadly, with the exception being James Cook University where research tends to be located in selected niche areas in science.

Table 1. Background data of the institutions

University	Founded	Under-graduates	Post-graduates	QS Ranking ^{#1}	THE Ranking ^{#2}
University of Queensland	1909	29,226	10,643	43	81
James Cook University	1970	12,093	3663	354	-
University of	1856	26,542	10,653	104	98

Maryland					
Purdue University	1869	31,145	7639	87	87

#1 – This is the Quacquarelli Symonds ranking: <http://www.qs.com/>

#2 – This is the Times Higher Education ranking:

<http://www.timeshighereducation.co.uk/world-university-rankings/>

University of Maryland

The University of Maryland case study is centred on initiatives to meet the needs of students pursuing majors in the biological sciences.

Initiation of change

Efforts to increase the quantitative training of biological sciences students arose from the changing landscape of scientific research. Many of the newly hired biological sciences faculty members reflected the increased quantitative emphasis of modern biology. Simultaneously there was a growing feeling among biological sciences faculty that students enrolled in upper-level courses did not show the degree of sophistication in quantitative reasoning that would be expected given the students' previous mathematical coursework. Also the department of mathematics had recruited a cohort of faculty members who were focused on biological problems.

Vision for change

Informal discussion among University of Maryland biological sciences faculty members resulted in a unified vision to infuse mathematics more deeply into the biology curriculum at all levels. The approach had three major components: (1) imbed basic mathematical content into introductory biology subjects for both majors and non-majors, (2) revise the mathematics sequence taken by biology students to be more biologically relevant, and (3) create an upper-level, quantitatively intensive subject in mathematical biology. The overarching goals of this coordinated approach were to help students appreciate the importance of mathematics and statistics for modern biology and allow students to more readily apply their quantitative knowledge to biological problems.

Implementing for change

The main strategy to imbed mathematical and statistical content into introductory biology subjects was to create the series of online modules called MathBench; see Nelson, et al. [11].

Shortly after initiating MathBench, faculty members in biological sciences began meeting with colleagues in mathematics to discuss the creation of a calculus sequence that would focus specifically on the types of mathematics that are most valuable for biologists and demonstrate the critical role of mathematics in understanding biological phenomena. To demonstrate the essential linkages between the fields of biology and mathematics, the subject has two problem-solving sessions per week in addition to lectures, one led by a mathematics teaching assistant and one led by a biology teaching assistant. The biology teaching assistants were also integral in the development of group problem-solving exercises that involve authentic biological problems. The subject was piloted with a small group of students in Spring 2008 and was instituted as a requirement for biology majors the following semester.

The third strategy to strengthen quantitative skills in biology students consisted of an upper-level Mathematical Biology subject that allowed students to develop sophisticated quantitative approaches to authentic biological problems. The chief developer of the new subject was a biology faculty member with formal graduate training in physics and research interests in computational neuroscience. Using a variety of tools (Excel and Matlab) and mathematical approaches (non-linear difference equations, eigenvector analysis, multi-

dimensional stability), the subject asks students to develop models to investigate important phenomena in diverse biological disciplines, including population dynamics, molecular evolution, phylogenetics, and infectious disease. The subject has been offered several times to very small numbers of students. It will soon reach a larger, broader audience as the capstone course of a new Honours program in Interdisciplinary Life Sciences, which enrolls approximately 75 students per year.

Key in the development and implementation of these reforms has been a reliance on an interdisciplinary faculty team consisting of biologists and mathematicians; within each discipline there were individuals who worked across disciplines as well as individuals with specific expertise in science education and curriculum development.

Evaluating the change

Multiple measures are used to evaluate the impact of these reforms, including pre- and post-tests of quantitative skill, student and faculty focus groups, surveying attitudes of graduating seniors, and tracking student grades. Students using MathBench show increases in critical quantitative skills that are independent of mathematical background and only slightly influenced by concurrent enrolment in a mathematics course [12]. A formal assessment of the impact of the revised calculus sequence on student performance in subsequent quantitatively-intensive coursework is underway. More informally, faculty members teaching the introductory physics sequence taken by biology students and those teaching upper-level biology subjects have noted a higher level of preparation of students since the implementation of this subject.

James Cook University

The initiative under discussion was delivered for the first time in the second half of 2010. It was introduced as one of a number of initiatives in curriculum reform, both across the Faculty of Science and Engineering and the university.

Initiation of change

The campus-wide curriculum reform agenda was seen as an opportunity to address deficiencies in student learning. In science, many students were observed to struggle when asked to use mathematics or statistics in a science context, demonstrating a lack of ability, willingness and confidence. Even more alarmingly, there was a perception amongst staff that the requirement to use mathematics or statistics in context was sufficiently unpalatable for some students to cause them to withdraw from the science program.

Vision for change

Discussions amongst the staff of the faculty revealed a belief that if the benefits of using mathematics and statistics to gain insight into problems in science could be demonstrated to students early in the program, they would engage with problems requiring these skills elsewhere in the science program in a more enthusiastic manner. As a result, a compulsory first year subject “Systems modelling and visualisation” was introduced. In order to demonstrate the relevance of mathematics and statistics to the entire science cohort it was anticipated that each of the three schools within the faculty would provide high-profile staff for prominent teaching roles in the subject.

Implementing for change

A statistician within the discipline of mathematics and statistics was chosen as the coordinator. She formed a committee consisting of representatives from information technology, biological sciences and physics to oversee the development of the subject. Her committee also had input from the Associate Dean Teaching and Learning who was

overseeing the curriculum renewal process across the faculty. A second biologist, prominent and highly regarded in her field and with extensive experience in the application of statistics was also brought in to the project. She and the coordinator of the subject were ultimately responsible for ensuring content alignment.

It is important to note that a significant number of students enter the subject through the university's alternate pathway in the Diploma of Science in which students study the prerequisite secondary school mathematics content in an accelerated form. Concerns over the mathematical ability of the cohort led the team to a case study approach in which six lectures at a time were devoted to a single study, the mathematics for which was not too extensive. The aim was to highlight methods of modelling and visualisation in each of the studies.

Case studies came from biological science, physics and climate change, reflecting the high-profile areas of research within the university that uses mathematics and/or statistics. There was considerable debate around the programming platform to be used, because many experienced staff believed that the hurdle associated with learning software is at least as challenging as learning the discipline content. Excel was chosen as the software tool because of the belief that students were likely to have seen it before.

Evaluating the change

The coordinating committee consulted with a science education specialist from within the university to develop ways of evaluating the effectiveness of the subject. This involved a questionnaire, follow-on focus group interviews and the perceptions of staff in follow-on subjects as to the ability of students in using mathematical and statistical thinking and reasoning in context. The results of this analysis will be published elsewhere.

The University of Queensland

The University of Queensland has introduced a range of activities directed at increasing the mathematical abilities of science students, in the contexts of each individual science discipline. This case study focuses on a capstone subject (BIOM 3200) for students in the biomedical science major of the Bachelor of Science, and how that subject builds on an introductory statistics subject.

Initiation of change

A recent review of the Bachelor of Science program identified a substantial lack of integration between mathematics, statistics and a range of science disciplines. As a result, a compulsory introductory statistics subject was introduced, taken by around 1000 science students each year. However, it became apparent that in many cases, the material covered in this subject was not reinforced in later subjects. Students typically lost confidence in their statistics ability, forgot how to apply their knowledge, and even came to believe that mathematics and statistics are unimportant in science. Faculty in biomedical science decided that it was essential to further build the quantitative skills of their students by integrating data analysis in the capstone subject. This was initiated by the subject coordinator, and included close collaboration with a discipline-based statistician.

Vision for change

The vision for the compulsory introductory subject was to cover statistical material regarded as essential for all science students. This subject was taught by a discipline-based statistician, and also included components of ethics, writing and quantitative communication. The goal for the capstone subject for approximately 250 students majoring in biomedical science was to provide an integrative learning experience bringing together biomedical science and statistics, and to further develop students' skills in ethics and communication. The importance of this integration was expressed by a senior biomedical science academic,

who identified that the highlight of the subject was the way in which the contextualised integration of science and mathematics would produce ‘very different and confident students’.

Implementing for change

The capstone subject was presented in three modules, with each module based on a biomedical problem. Students worked in groups, assisted by tutors, building on prior disciplinary learning and connecting biomedical science with statistics, ethics and communication to enhancing their biomedical science knowledge. In the data analysis activities, each group of students was provided with a set of raw data derived from published biomedical science research by the statistician. Students undertook analysis, interpretation, presentation of results and report writing, treating the data as an integral component of the process by which the underlying research problems were analysed, interpreted, understood and communicated to a diverse audience.

It rapidly became apparent that biomedical science tutors would be unable to facilitate understanding of the statistics, because they lacked the insight or expertise in using and explaining statistics. To assist in overcoming this challenge, two tutors were present in each class, one from biomedical science and the other with expertise in statistics. To further facilitate understanding, the discipline-based statistician provided the following additional support:

- students were given a series of application-based lectures before they commenced data analysis;
- biomedical science tutors received a training session for each module; and
- the discipline-based statistics tutors attended a workshop on what they could reasonably expect students to know, and the most appropriate approaches to use.

Evaluating the change

The capstone subject was evaluated using a tailor-made survey to explore the effectiveness of the initiative. Questions focused on whether students increased their confidence in data analysis, improved their quantitative skills, and developed new insights into how to approach scientific data analysis. The majority of the students appreciated the value of disciplinary and interdisciplinary integration that occurred in the subject, and also identified that there was integration of their prior learning, from both biomedical science and statistics.

Purdue University

At Purdue University, an understanding of the synergy between statistics and science is achieved through a process of writing tasks and peer review.

Initiation of change

As with other case studies presented in this report, faculty members in biology recognized that the ability to apply mathematical and statistical thinking is becoming increasingly important because of the changing nature of scientific research. At the same time, there was evidence that students enrolling in bioscience programs were often substantially underprepared in understanding and appreciating the roles that probability and statistics play in dealing with the inherent variation of biological systems.

Vision for change

Despite a range of previous initiatives aimed at developing links between mathematics, statistics and biology, a significant gap in required knowledge was seen to remain. Several recent efforts have been explicitly designed by faculty to further prepare bioscience students to apply mathematical and statistical reasoning and also describe natural phenomena, and

validate scientific knowledge.

Implementing for change

A recent National Science Foundation award entitled *Teaching Ethical, Experimental, and Quantitative (TEEQ) Biology through Problem-Based Writing with Peer Review* was initiated at Purdue University by collaboration between a biologist and a statistician in response to the need for students to understand the role of probability and statistics in analyzing variation in biological systems. The project adapted the Calibrated Peer Review (CPR) process, developed at the University of California at Los Angeles, as a mechanism for increasing student understanding of experimental methods and quantitative approaches in biology.

The process works as follows. Each student is presented with a contextualized problem with a substantial quantitative basis, and then asked to write an analysis and discussion. Students are then given guiding questions to build their competence in the scientific review process. Following this, each student receives three peer documents to review using the guiding questions and then assign a score. Finally, the student undertakes a review of their own work. The student's grade is based on both their own writing and their peer reviewing.

New quantitative problem-based writing assignments with peer review have been incorporated into an introductory subject, Biology 131, to help students connect what they learn to both current and historical research endeavors. Biological problems with writing assignments for peer review are also being incorporated in the Statistics 301 subject (a subject also catering for students outside science) to help students understand how new knowledge accumulates in the biosciences. Students also consider what ethical constraints, such as predictions of the expected number of animals for a research study, must be considered. The project targets more than 1300 students annually.

Evaluating the change

A Participant Perception Inventory was developed as a project evaluation tool. This is a questionnaire designed to measure student perception of their own knowledge (cognitive dimension), experience (behavioral dimension), and confidence (affective dimension) about ethical, experimental, and quantitative aspects of research. Results show that the CPR assignments provide a tested method to help students learn by lowering barriers to primary literature. Student responses were examined using a factor analysis method to determine groups of questions that are answered by students in a correlated manner, thus indicating aspects of student thinking that are closely linked. The factor analysis scores were also used to inform teaching by identifying categories that may need to be taught in an explicit manner. For example, results suggest that visualization associated with research is an important skill that may need to be more explicitly addressed.

Discussion and Conclusion

Analysis of the change process

In each of the initiatives presented, the initiation and vision for change was driven by the need for student understanding of some aspect of science to be enhanced by the ability to apply mathematical or statistical thinking or reasoning to that scientific context. In the two cases from the USA, faculty in biology were the initiators, while in the two cases from Australia science faculty were involved in the initiation, but broader teaching and learning agenda also facilitated their initiation.

Considerable variety exists in implementation. In comparing the two USA studies, the initiation for change was almost identical; however the vision and implementation provide a significant contrast. In the Maryland case, a calculus subject for biology students was

developed in partnership between faculty in mathematics and biology. Students were exposed to teaching assistants from both mathematics and biology. At Purdue the partnership between a biologist and statistician resulted in a problem-based writing and review task, embedded within a statistics subject catering for the needs of students outside biology as well as within. Thus the links between statistics and science were developed as a goal within a general statistics subject.

Similarly, in comparing the two Australian studies, the initiation for implementing core subjects in the science program was similar; however the specific subjects introduced at the two institutions were quite different in their approach to developing the skills of the students. These contrasts serve to highlight the variety of options available to those grappling with the issue of how to embed the development of mathematical and statistical thinking within science contexts.

In each study significant steps have been taken to evaluate the effectiveness of the changes implemented. While they show considerable success in some areas, there are challenges associated with the interdisciplinary nature of these initiatives. Undoubtedly these challenges were anticipated, as evidenced by the pairing of tutors from differing backgrounds in the Maryland calculus course and the University of Queensland capstone subject. Research around how best to develop relationships between teaching staff of differing discipline backgrounds is needed to overcome these challenges.

The role of mathematicians and statisticians

In all of these initiatives, it is particularly interesting to observe the degree of importance placed on cooperation between mathematicians, statisticians and scientists. It is clear that the four projects presented here involved collaboration across discipline boundaries, some in the formulation of resources and others in the day-to-day teaching. One factor that hasn't been made explicit to this point is the level of support in terms of funding required to allow them to succeed – each of the initiatives involved substantial funding that probably cannot routinely be made available across large numbers of institutions. Whether these collaborations would occur in the absence of funding is a significant question to ponder.

The four case studies present instances in which the default position (science students needing to deduce the links between mathematics and science for themselves) was reconceptualised to better achieve the intended outcomes of contextualising mathematical learning for science students. While the initiation to rethink the default position varied across the case studies, all implementations involved collaboration and cooperation across traditional discipline boundaries, with mathematicians, statisticians and science faculty all teaching in the broader context of quantitative science. The case studies demonstrate that successful collaboration across disciplines is possible and can improve the mathematical skills of science students. This raises broader questions as we move beyond a “single subject” view of teaching the application mathematics to science students: how do we better link the “pure” mathematical knowledge gained in mathematics-rich courses to the contextualised mathematics taught in science-rich courses? How do we create sustainable frameworks that allow for interdisciplinary collaborations with a view to building the mathematical skills of science students across all levels of the degree program? Finally, how do we gather evidence to inform the ongoing efforts of mathematicians and scientists as they work together to achieve these interdisciplinary learning outcomes?

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Teaching and learning proof-writing in linear algebra

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Abstract

A teaching initiative, in which we attempted to help students prove that a set of vectors is closed under scalar multiplication and addition, is described. By means of a template illustrating closure proofs in three different vector spaces, students' attention was drawn to the underlying structure of such proofs, and the commonality of procedures in each of these cases was made explicit. Following exposure to the template, students were asked to write their own closure proofs for a given set of vectors in \mathbb{R}^3 and were also asked to describe the vectors in that set in their own words. Their attempts at the closure proofs and the way in which they conceptualised the given set of vectors were analysed.

1 Introduction

There is a considerable body of literature relating to the difficulties students encounter in learning linear algebra, and the reasons for those difficulties. In a previous paper (Britton and Henderson [1]) we provided a short summary of this literature, and noted that almost all researchers mention students' inexperience with proofs as a contributing factor.

The difficulties students find in constructing a proof are also well-documented. (See, for example, Dreyfus [2], Easdown [3], Moore [4].) It certainly seems to be the case that the word "proof" strikes fear into the hearts of many students, and it is not uncommon to hear students assert that they "can't do proofs". The problem is compounded by the fact that it is often during a course in linear algebra that students are asked for the first time to write a mathematical proof, and are assessed on their ability to do so. Students struggling to understand the concepts of linear algebra are at the same time struggling with the task of writing proofs. Furthermore, as Moore [4] points out, in many universities no explicit instruction in how to write proofs is given and in such situations students are expected to pick up this skill by reading proofs written by their lecturer. We should not be surprised when they find this a difficult task.

Some of the first proofs that students encounter in linear algebra deal with proving that particular sets are subspaces of given vector spaces, and this involves proving that sets are closed under addition and scalar multiplication. Writing a valid proof that a set is closed may well seem a relatively simple task, and one that could reasonably be viewed by instructors as procedural in nature. But it is not, of course, simple for students, and more than just following a procedure is required. In order to be successful at the task, students need to have an understanding of the concept of closure, to be able to make sense of set notation, to know how to write an arbitrary vector in a set, and to set out their statements logically. Nevertheless, the structure of a valid closure proof remains the same for different sets, and the procedures to be followed do not change. This suggests that students might be helped to write a closure proof by highlighting the structure of the proof and making the procedures highly visible. The possibility of a teaching initiative of this nature was foreshadowed in our previous paper [1]. In the present study the initiative was implemented, and some outcomes examined.

We were also interested in investigating the way in which students conceptualised the given set of vectors, and whether there was any relationship between the nature of their conceptualisation and the level of success in writing a closure proof. Hence, when we asked them to write

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two closure proofs as an exercise, we also asked them to describe how they conceptualised the given set of vectors.

In the next section, we describe the teaching initiative in detail, and then give an analysis of the results of the exercise.

2 The teaching initiative

The students involved in this study were enrolled in a second year service course in linear algebra and vector calculus. The course is taken by engineering and science students, and had an enrolment of 640 students in 2011. Both authors have been involved in teaching this course over many years, and one of the authors was the sole lecturer for the linear algebra section at the time of the study. The students are not aiming to be specialists in mathematics and the course is designed to give them a good understanding of and facility with the material with less emphasis on abstraction and a lower level of sophistication than traditional linear algebra courses designed for advanced level students.

The linear algebra section of the course introduces abstract vector spaces, span, linear independence, subspaces, null and column spaces of matrices and diagonalisation. It includes applications such as solving linear recurrence relations, and solving systems of linear differential equations. (This material builds on content covered in a first year linear algebra course, during which students are introduced to geometric vectors, matrices, Gaussian elimination, determinants, eigenvalues and eigenvectors.)

Abstract vector spaces, and the concept of closure of a set, are introduced in the first week, and subspaces are introduced in the second week. The majority of examples of vector spaces and subspaces given in lectures involve vectors in \mathbb{R}^2 or \mathbb{R}^3 . (In this course \mathbb{R}^n is defined as the set of all $n \times 1$ matrices (column vectors) with real entries.) However, since this is a second course in linear algebra, we would expect students to be able to move beyond the idea of a geometric vector to the more abstract concept, and so other vector spaces are also discussed. For example, the vector spaces P_n , the set of polynomials of degree less than or equal to n , and F , the set of real-valued functions defined on \mathbb{R} , are introduced, and examples in \mathbb{R}^n for $n > 3$ are provided. Geometric interpretations of the subspaces of \mathbb{R}^3 are given, but we are mindful of the strong claim made by Harel [5, pp. 612, 613] that geometric illustrations can hinder the learning of the true concepts of linear algebra. He states :

When geometry is introduced before the concept has been formed, the students view the geometry as the raw material to be studied. They remain, as a result, in the restricted world of geometric vectors, and do not move up to the general case.the instructor sees how the geometric situation is isomorphic to the algebraic one[but] the student does not share this important insight.

We therefore make a point of emphasising the distinction between a vector in \mathbb{R}^3 and a point (x, y, z) in 3-dimensional Cartesian space, and that the subspaces of \mathbb{R}^3 may be interpreted geometrically as planes or lines through the origin in 3-dimensional Cartesian space only by associating each vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with the corresponding point (x, y, z) .

During lectures in the second week, students were given a hand-out whose contents are reproduced below. It included a step-by-step procedure for proving closure, and examples of proofs for three different sets. Students were advised that the hand-out could be used as a template to assist in constructing their own proofs.

The first part of the hand-out contained the following step-by-step instructions, which we hoped would help students at least make a start on a proof. (Moore [4] found seven major

sources of difficulty for students in writing proofs, one of which was: “The students did not know how to begin proofs”.)

To prove that a set S is closed under scalar multiplication you should do the following:
Step 1: Write down these two sentences:
 Let $\mathbf{u} = \dots\dots$ be an element in S . Let k be a real number.
Step 2: Show that $k\mathbf{u}$ is an element of S .
 To prove that a set S is closed under addition you should do the following:
Step 1: Write down this sentence:
 Let $\mathbf{u} = \dots\dots$ and $\mathbf{v} = \dots\dots$ be elements in S .
Step 2: Show that $\mathbf{u} + \mathbf{v}$ is an element of S .
 The vectors that replace the dots $\dots\dots$ in Step 1 must be arbitrary, or generic, elements of S , as illustrated in Step 1 in the examples overleaf. Then, in Step 2 of the proof for scalar multiplication, you need to show that $k\mathbf{u}$ has the same form as your arbitrary vector, and in Step 2 of the proof for addition, you need to show that $\mathbf{u} + \mathbf{v}$ has the same form as your arbitrary vector(s).
 So before you start your proof, you need to think about how to write such a vector. It is a good idea to start by describing the set in your own words. It also helps to write down some particular vectors that belong to the set, and some particular vectors that do not belong to the set.

Underneath these step-by-step instructions, three different sets were given, listed in three separate columns.

Let $V =$ $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 2x \right\}$	Let $S =$ $\{f \in \mathbb{F} \mid f(1) = 0\}$	Let $T =$ $\left\{ s \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$
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The next section gave various different descriptions of the sets. Our experience is that many students find difficulty in writing an appropriate arbitrary vector to use in their proofs. We hoped that students would be assisted in the process of finding an arbitrary vector for a particular set if they were encouraged to think about various ways of describing the vectors in the set. Again, three columns were used, corresponding to the sets above.

By associating each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ with the point (x, y) in 2-dimensional space, the set can be interpreted as a straight line in 2-dimensional space. The vectors in V all have the form $\begin{pmatrix} x \\ 2x \end{pmatrix}$. The vectors in V are scalar multiples of the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.	The elements of S are all the functions in \mathbb{F} that map the number 1 to the number 0. The functions in S are such that their graphs pass through the point $(1, 0)$.	The vectors in T are linear combinations of the vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$. By associating each vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with the point (x, y, z) in 3-dimensional space, the set can be interpreted as a plane in 3-dimensional space.
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In order to further assist students in conceptualising precisely the types of vectors that belong to each set, the next section of the hand-out gave several examples of particular vectors that belong to each of V , S and T , and several of particular vectors that belong to \mathbb{R}^2 , \mathbb{F} , \mathbb{R}^3 but not to V , S , T (respectively). (We have not included these examples here.)

The final section of the hand-out contained a proof of closure under scalar multiplication, and a proof of closure under addition, for each of the sets. For brevity, we have included here only the proofs of scalar multiplication. The three-column format was maintained for this section, and the steps described in the first section of the hand-out were highlighted.

<p>Step 1: Let $\mathbf{u} = \begin{pmatrix} t \\ 2t \end{pmatrix}$ be an element of V, where $t \in \mathbb{R}$. Let k be a real number.</p> <p>Step 2: Now we have to show that $k\mathbf{u} \in V$. $k\mathbf{u} = k \begin{pmatrix} t \\ 2t \end{pmatrix} = \begin{pmatrix} kt \\ 2kt \end{pmatrix}$. The second component of $k\mathbf{u}$ is twice its first component. Hence, $k\mathbf{u} \in V$ and V is closed under scalar multiplication.</p>	<p>Step 1: Let f be an element of S, so that $f(1) = 0$. Let k be a real number.</p> <p>Step 2: Now we have to show that $kf \in S$. $(kf)(1) = k \times f(1) = k \times 0 = 0$. That is, $(kf)(1) = 0$ and so $kf \in S$ and S is closed under scalar multiplication.</p>	<p>Step 1: Let $\mathbf{u} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ be an element of T, where a and b are real numbers. Let k be a real number.</p> <p>Step 2: Now we have to show that $k\mathbf{u} \in T$. $k\mathbf{u} = k \left(a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) = (ka) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (kb) \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Since ka and kb are real numbers, $k\mathbf{u}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. So $k\mathbf{u} \in T$ and T is closed under scalar multiplication.</p>
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During tutorials in week 4, students were asked to complete the following exercise:

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 5x + 2y - z = 0 \right\}$.

1. Describe the elements of the set V in your own words.
2. Write down 2 particular vectors that belong to V , and 2 that do not.
3. Prove that V is closed under scalar multiplication.
4. Prove that V is closed under addition.

Students were allowed to refer to any materials during this exercise, including the hand-out described above, and they were also allowed to talk to other students while completing the task. Tutors had been asked not to give assistance, however.

A total of 497 responses were collected, and analysed.

3 Classification of the student responses

Each response received two different classifications, the first on the basis of the description of the elements of the set V written in their own words (part 1 of the exercise) and the second on the basis of their attempts to prove closure of V under scalar multiplication and addition (parts 3 and 4 of the exercise). During the classification process, checks were made on the particular vectors nominated by students as belonging to or not belonging to V (part 2 of the exercise). Since virtually all students were able to complete part 2 successfully, we did not further consider responses to this part.

3.1 Describing the elements of V

Part 1 of the exercise was designed to give insight into the students' degree of understanding of set notation and the ability to translate a sentence given mathematically into a more informal representation. A variety of responses was expected, but it was thought that many students would make a connection between the algebraic definition of the set V and its corresponding geometric object, namely the plane through the origin in 3-dimensional space with Cartesian equation $5x + 2y - z = 0$. Approximately one hundred responses to part 1 of the exercise, chosen randomly from the full set of 497, were read independently by each author. After discussion, a classification framework involving four categories was agreed upon. All 497 responses were then read and classified by each author (again independently) under one of the four categories R, T, P, T+P listed below. Some allowances were made for bad expression and grammar as the class contains many students from a non-English speaking background. Any initial differences between authors in the classification of particular responses were resolved by discussion.

Category R : This category contains responses which exhibit fundamental errors in understanding of set notation and the meaning of symbols, or a totally incorrect interpretation of the set V . Examples of responses classified R are given below. (Note: All examples of students' work in this article are reproduced verbatim, including any grammatical or spelling errors.)

1. The set of V is the equation $5x + 2y - z = 0$ such that x, y and z are an element of R^3 and are real numbers.
2. V is a non-empty vector which passes through the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and satisfies equation $5x + 2y - z = 0$.
3. The elements of V are a line passing through $(0, 0, 0)$.

Category T : Responses in this category were essentially translations of the symbolic definition of elements of V into words, without any mention of a plane. Less than perfect translations were admitted as long as there were no serious errors of the type found in category R. Examples of responses classified T are given below.

1. V is a vector in 3-D with coordinates x, y, z where x, y, z satisfy that $5x + 2y - z = 0$.
2. V is the set of all elements $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ contained in R^3 that satisfy the equation $5x + 2y - z = 0$.
3. V is the set of any vectors with 1 column and three rows, whose elements, when put into the equation $5x + 2y - z$ give a result equal to zero.

Category P : Students in this group expressed their understanding of the elements in V essentially in terms of the corresponding plane, sometimes with minimal information and sometimes with full information. Examples of responses classified P are given below.

1. The elements in the set V describes vectors in a plane.
2. The geometric description of the elements in V is a plane which passes through the origin.
3. V is the set of all points which lie on the plane with equation $5x + 2y - z = 0$.

Category T+P : Responses here are characterised by an approach involving elements of a verbal translation in combination with a geometric interpretation. Examples of responses classified as T + P are given below.

1. By associating each vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with the point (x, y, z) in 3-dimensional space, the set can be interpreted as a plane in 3-dimensional space. The vectors are all of the form $\begin{pmatrix} x \\ y \\ 5x+2y \end{pmatrix}$.
2. The elements of set V are all vectors in R^3 , however set V contains only those vectors in R^3

such that fall in the plane described by the equation $5x + 2y - z = 0$.

3. A vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that exists in R^3 such that the equation $5x + 2y - z = 0$ is satisfied. i.e. x, y, z are points on plane with cartesian equation $5x + 2y - z = 0$.

3.2 Two closure proofs

A random selection of responses to the closure tasks was read independently by each author and a classification framework involving five levels was agreed upon. The full set of 497 responses was then read and classified by each author independently and any differences resolved by discussion. Responses were classified as Level 1, 2, 3, 4 or 5. Levels 1 and 2 contained all responses in which students were unable to nominate appropriate arbitrary vectors and were thus unable to proceed to valid closure proofs. Levels 3, 4 and 5 contained all responses where an appropriate choice of arbitrary vectors and an arbitrary scalar were made, leading to the possibility of constructing valid closure proofs. (Students classified at these levels who used generic R^3 vectors as the arbitrary vectors were required to have shown awareness that the components satisfy the given condition.) These three levels represent work in order of mathematical merit, with level 5 being the highest quality.

Level 1 : Inappropriate arbitrary vectors were chosen, leading to invalid closure proofs. Some responses revealed elements of understanding of the closure concept, but even in these cases what was written was too far from the question at hand to be considered appropriate. The following example illustrates a response classified as level 1. (Note that this is the complete response – the symbols used were not defined by the student.)

For scalar multiplication: Let $v = 5v_1 + 2v_2 + v_3$ and k be some scalar R^3 such that v and $k \in V$. $kv = k(5v_1 + 2v_2 + v_3) = 5(kv_1) + 2(kv_2) + kv_3 = (kv)$.

Therefore $kv \in V$ and V is closed under scalar multiplication.

For addition: Let $u = 5u_1 + 2u_2 + u_3$ such that v and $u \in V$.

$$v + u = 5v_1 + 2v_2 + v_3 + 5u_1 + 2u_2 + u_3 = 5(v_1 + u_1) + 2(v_2 + u_2) + (v_3 + u_3).$$

Therefore $v + u \in V$.

Level 2 : Particular (numerical) non-zero vectors in V were used instead of arbitrary vectors, leading to invalid proofs. For example:

For scalar multiplication: Let $k = 2$. $\begin{pmatrix} 2 \\ 0 \\ 10 \end{pmatrix} \times k = \begin{pmatrix} 4 \\ 0 \\ 20 \end{pmatrix}$.

$20 + 0 - 20 = 0$ therefore closed under scalar multiplication.

For addition: $\begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 17 \end{pmatrix}$. $15 + 2 - 17 = 0$ therefore closed under addition.

Level 3 : Appropriate arbitrary vectors and an arbitrary scalar were given, but in both closure tasks the attempts at proof were unsatisfactory. This level included work where statements did not follow in logical order. For example:

For scalar multiplication: Take vectors $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ as vectors in V and k is a real number.

$$k \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix} = 5kx_1 + 2ky_1 - kz_1 = 0, \quad k(5x_1 + 2y_1 - z_1) = 0$$

$$5x_1 + 2y_1 - z_1 = 0$$

which satisfies $5x + 2y - z = 0$ and therefore is closed under scalar multiplication.

For addition: Take vectors $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ as vectors in V . $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} =$
 $5(x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2) = 0$. Let $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = 5x_3 + 2y_3 - z_3 = 0$ which

satisfies $5x + 2y - z = 0$ and is therefore closed under addition.

Level 4: Appropriate arbitrary vectors and an arbitrary scalar were given and only one of the two closure proofs was correct. (In the example which follows, we did not accept the proof for scalar multiplication.)

For scalar multiplication: Let $\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V$ and $k \in \mathbb{R}$.

Therefore $5a + 2b - c = 0 \implies \mathbf{u} = \begin{pmatrix} a \\ b \\ 5a+2b \end{pmatrix}$. $k\mathbf{u} = \begin{pmatrix} ka \\ kb \\ 5ka+2kb \end{pmatrix} \implies \begin{pmatrix} ka \\ kb \\ k(5a+2b) \end{pmatrix} \in V$.

Therefore V is closed under scalar multiplication.

For addition: Let $\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \in V$.

Therefore $5a + 2b = c$, $5d + 2e = f$ and

$\mathbf{u} + \mathbf{w} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ 5a+2b+2d+2e \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ 5(a+d)+2(b+e) \end{pmatrix} \in V$. Therefore V is closed under addition.

Level 5: Appropriate arbitrary vectors and an arbitrary scalar were given and both closure proofs were correct. For example:

For scalar multiplication: Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V$. $k \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$.

Because $5a + 2b - c = 0$, therefore $5ka + 2kb - kc = k(5a + 2b - c) = 0$. Therefore closed under scalar multiplication.

For addition: Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V$, $\begin{pmatrix} d \\ e \\ f \end{pmatrix} \in V$. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$.

$5(a+d) + 2(b+e) - (c+f) = (5a + 2b - c) + (5d + 2e - f) = 0 + 0 = 0$. Therefore closed under addition.

The example of a level 5 response given here demonstrates that what was considered an acceptable closure proof in this exercise is less rigorous than would be expected of a student studying mathematics at the advanced level. Latitude was given here because of the relatively short time since the concepts had been introduced and also because of the nature of this particular cohort of students, previously described in section 2.

4 Results of the study

4.1 Relationship between students' conceptualisation of V and the quality of the closure proof.

The numbers in each of the five levels for each of the four categories of description of elements of V , and overall, are given in Table 1.

	Level 1	Level 2	Level 3	Level 4	Level 5
Category R (n=75)	25	4	24	10	12
Category T (n=181)	53	19	25	37	47
Category P (n=189)	53	11	24	30	71
Category T+P (n=52)	6	6	7	12	21
Total (n=497)	137	40	80	89	151

Table 1: Distribution of responses by category and level

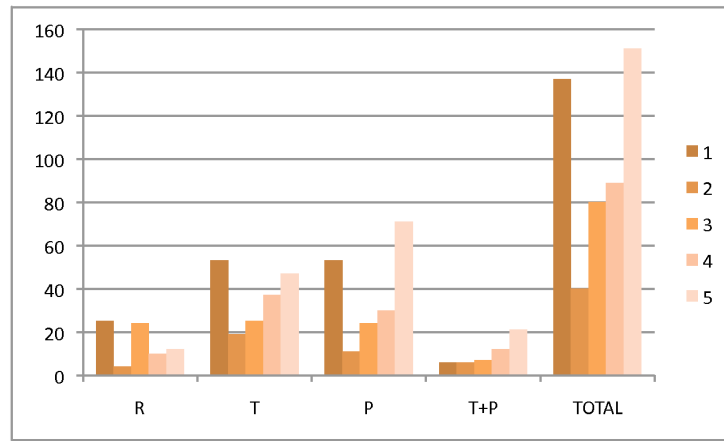


Figure 1: Distribution of levels on closure tasks by category, and overall

These data are also displayed in Figure 1 where it is a little easier to notice some of the differences between the categories. All categories except T+P appear to have relatively large numbers of level 1 students (unable to write down arbitrary vectors), while category R also has relatively large numbers of level 3 students (unable to make proper use of appropriate arbitrary vectors and hence failing to construct closure proofs).

Only those students who correctly identified arbitrary vectors (that is, those classified as level 3, 4 or 5) were in a position to construct a valid closure proof, and the greatest likelihood of this happening occurs in category T+P, in which 77% of students successfully completed this preliminary task. Table 2 displays these data.

	% able to identify appropriate arbitrary vectors
Category R (n=75)	61%
Category T (n=181)	60%
Category P (n=189)	66%
Category T+P (n=52)	77%
Total (n=497)	64%

Table 2: Percentage of students correctly identifying arbitrary vectors

Table 3 shows that students in category T+P did better than those in any other category as measured by the percentage classified as level 4 or 5 (achieving success in proving closure in either one or both of scalar multiplication and addition). This category also had the lowest percentage classified as level 1.

Table 3 also shows that 29% of students in category R were classified as level 4 or 5 on the closure proofs. This would appear at first glance to be a perplexingly large percentage of this category. However, the majority of these 22 students were classified as R either because they stated that the points satisfying $5x + 2y - z = 0$ lie on a line rather than a plane, or because they gave a correct but irrelevant response (such as "The set V is non-empty"), or because their poor capacity for English expression produced garbled responses. Indeed, eight of these students identified the set as corresponding to a plane, but were unable to write a meaningful

	% at Level 1	% at Level 4 or 5
Category R (n=75)	33%	29%
Category T (n=181)	29%	46%
Category P (n=189)	27%	53%
Category T+P (n=52)	12%	63%
Total (n=497)	28%	48%

Table 3: Percentage of students at Level 1 and at Level 4 or 5

description. Deficiencies such as these in the first part of the exercise would not be expected to impede the ability to write at least one acceptable closure proof.

Overall, however, the data suggest that there is a link between the manner in which the set of vectors is conceptualised and the degree of success in constructing closure proofs. Students in category T+P demonstrated they could understand and de-code the definition of the set V and conceptualise the associated geometric object. It is not unreasonable to suppose that these students had a fuller and more nuanced understanding of the objects they were dealing with than students in categories R, T and P whose descriptions were either in error or more narrowly focussed. It appears that the students giving the fullest description of the vectors tended, on the whole, to be the most successful in writing closure proofs.

4.2 An examination of the closure proofs

As shown in Table 3 above, almost half the students were able to write at least one satisfactory closure proof. Of those students unable to do so, 80 were able to identify appropriate arbitrary vectors but were not able to progress further, while 40 chose particular vectors, and the remainder attempted to write an arbitrary vector but were unsuccessful. The most common errors made by this latter group were the use of the expression $5x + 2y - z$ as a vector, and the use of an incorrect parametric form (for example, $\begin{pmatrix} 5x \\ 2y \\ 5x+2y \end{pmatrix}$).

In our examination of the responses from students who were able to identify an appropriate arbitrary vector, and hence were in a position to construct a proof, two interesting points arose.

The first relates to the choice of an arbitrary vector to begin the proof. For the set $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 5x + 2y - z = 0 \right\}$ given in the exercise, there are basically two ways to write down an arbitrary vector belonging to the set. An arbitrary vector in \mathbb{R}^3 subject to the defining equation can be used, as in: "Let $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $5x + 2y - z = 0$, be a vector in V ." Alternately, the condition $5x + 2y - z = 0$ can be incorporated into the vector, so that a parametric form of the vectors in V is obtained, as in: "Let $u = \begin{pmatrix} x \\ y \\ 5x+2y \end{pmatrix}$ be a vector in V ." (The equation $5x + 2y - z = 0$ was deliberately chosen so that it was particularly easy to obtain this form.) Students who used this second alternative were markedly more successful in completing a valid proof than were those who used the first alternative. Of the 80 students who were able to write an appropriate arbitrary vector but unable to write a satisfactory proof (that is, those classified as Level 3), only 5 had used the parametric form. In other words, only 5 students writing an appropriate arbitrary vector in parametric form were unable to go on and write at least one satisfactory closure proof, while 75 students using the defining equation to describe an arbitrary vector were unable to do so. This is not, perhaps, surprising. Not only is it arguably easier to write a correct proof using the parametric form, but one might conjecture that students using this form had given more thought to the exercise than those who had simply used the

definition provided. Such a conjecture is supported by the figures in the table below, which show that of those students classified as level 5, category T+P had the highest percentage using the parametric form.

	% of Level 5 students using parametric form
Category R	42%
Category T	49%
Category P	48%
Category T+P	71%

Table 4: Percentage of students classified as Level 5 using the parametric form

Secondly, we were somewhat surprised to find that students were overwhelmingly more successful in proving closure under addition than they were in proving closure under scalar multiplication. Of the 89 students who were classified as level 4 (that is, those who had written exactly one correct closure proof), 71 had written a correct proof or closure under addition, and 18 a correct proof or closure under scalar multiplication. It would appear that the necessity to introduce another symbol (generally k , for a scalar) in the proof of closure under scalar multiplication was a confounding factor for many students.

It would be worthwhile keeping in mind both the points raised here when teaching this material.

5 Summary and discussion

The teaching initiative described in this study was designed with several purposes in mind. The major purpose was to provide students with material that might assist them in constructing a valid proof that a given set is closed. This particular proof was chosen because it appears early on in our second year linear algebra course, and because it was relatively easy to highlight the structure of a closure proof in the template we provided. Secondary purposes included encouraging students to interpret set notation in various ways, and to improve their ability to read and understand proofs. (Cowen [6, p. 50] makes the point that we should make a conscious effort to "... teach our students to read and understand mathematics.")

We should emphasise that, while the template used in this study is quite detailed and prescriptive, we are not suggesting that students should be expected to write proofs on autopilot. On the contrary, even structured assistance of this nature requires students to pay close attention to the fine detail of the definitions and objects they are manipulating. Nor are we suggesting that the provision of such a template would be universally appropriate. Nevertheless, students can, and should, be explicitly helped to learn how to write proofs. Indeed, Malek and Movshovitz-Hadar [7] have reported that students can be helped to construct complex proofs through exposure to "transparent pseudo-proofs (TTPs)". (They define complex proofs as those which require new ideas and connections to be made, and a TTP as "... a proof of a particular case which is small enough to serve as a concrete example, yet large enough to be considered as a non-specific representative of the flow of arguments in the proof of the general case." [7, p. 35]) They claim that, after studying a TTP that had been constructed to illuminate the proof of a particular theorem, students were able to use the TTP as a springboard to write a valid proof in the general case, and to explain the idea behind their proof orally.

The template approach may well provide a similar springboard for students as they learn

to write a simple proof. In this study no attempt was made to quantify the effect of using the template. Rather, we wanted to observe the quality of the proofs written by students, and to determine whether or not any further implications for teaching arose. The small percentage of students able to write two valid closure proofs was disappointing, although not entirely unexpected. It should be noted that students were asked to do the proof-writing task without notice, and had no opportunity to prepare as they would normally have done in advance of an examination. However, our view is that drawing explicit attention to the structure of closure proofs in this way has the potential to make a difference in several areas.

Firstly, it enables students to see a unified structure in which to view all closure proofs, which may help them in the transition process from the specific to the general as they learn more about different vector spaces and subspaces. Secondly, the need for an appropriate choice of arbitrary vector assumes a central role and this in turn reminds students of the necessity for a clear understanding of the vectors in the given set. The study demonstrated that students with more complete understanding of the nature of the set whose closure was to be proved were more likely to be successful in the task. Thirdly, using the template shows students that there are frameworks for practical support in proof-writing, giving them help in starting a proof and showing them that their lecturers understand that proof-writing is a learned skill which takes time to acquire. Given the apprehension with which many students historically approach this task, this positive aspect of the template approach may go some way to mitigating their fear of proofs.

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Exploring Learning Design in Tertiary Mathematics

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This research is motivated by challenges in the design of learning materials and their delivery through a blend of traditional and e-learning environments. The initial focus addresses the question “what is learning design?” Definitions of learning design have changed in response to educator’s attempts to implement learning designs in their subjects and in response to their attempts to articulate how to design for better learning. Building on Herrington & Oliver’s (2002) model which directs attention to resources, tasks and supports in designing online learning environments, this study applies a learning design model to an introductory subject (MATH151). The underlying question asked is “how can resources and tasks best be combined in e-learning to support students in learning?” This has led to the creation of a learning design for MATH 151 with emphasis on visual components rather than written descriptions. One question addressed in the design process is “how can the temporal aspects of the subject be linked to resources?” The design process has generated questions about the nature and potential benefits of using designs to improve clarify concepts and to understand the learning process. An extension of these questions involves asking “how can students be better engaged in the learning process?”

Keywords: Visual learning design, pedagogical, resource based learning, e-learning, blended learning.

Introduction

The development of technology has changed pedagogical approaches to teaching and learning at the tertiary level. This change has come with the use of information and communications to achieve desired learning outcomes (Boud & Prosser, 2001). The emergence of new learning technologies has coincided with a growing awareness and recognition of learning theories that suggest there are many problems and inefficiencies with conventional forms of teaching (Oliver & Herrington, 2003). Universities have welcomed the capability of online systems to promote access to information and services for students irrespective of location, or computing skills. Whether e-learning in itself provides a solution for many of the problems facing institutions of higher education is debatable (Oliver, 2001). This research was prompted by the challenges posed by an increased demand for information and interaction in university subject websites including, ensuring high quality resources are provided. It also addresses the challenge of increasing lecturers’ knowledge and awareness as to how to best make resources available to support university students in mathematics courses. The background for this paper is the area of e-learning and specifically e-learning of mathematics.

In the search for answers to the questions “how can resources and tasks best be combined in e-learning to support students in learning?”, and “how can students be better engaged in the learning process?” our initial focus is to understand what constitutes the elements of a good learning design. Definitions of learning design, their purpose and components have changed in response to educators implementing learning designs in their subjects and attempts to articulate how to design for better learning.

Definitions of Learning Design

The term “learning design” is used in different ways. According to Oliver (1999) learning design should focus on tasks, supports and resources. These “provide a strong framework for instructional design, and highlight the importance of planning specific roles for learners, the teacher and the technology in the learning environment” (p. 343). With this framework it is possible for students to access resources in a multiplicity of ways, choosing which materials to use and how to use them. Oliver (2001) and Oliver and Herrington (2003, p. 13) suggest the following design stages as the basis of effective and efficient approaches to design:

- *Design of learning activities.* Design the activities and interactions for engaging and directing the learner in the process of knowledge acquisition and development of understanding.
- *Provision of learning resources.* Develop and provide the content, information and resources upon which learning is based given learner interaction. Resources are needed by the learner to successfully complete the set tasks and to facilitate the scaffolding and guidance.
- *Design of learning supports.* Learning supports are the strategies planned to engage learners with the tasks and to enable them to complete tasks. This includes scaffolds, encouragements, assistance and connections used to support learning by providing guidance and feedback in the learning process. There three stages overlap and intersect as represented in Figure 1.

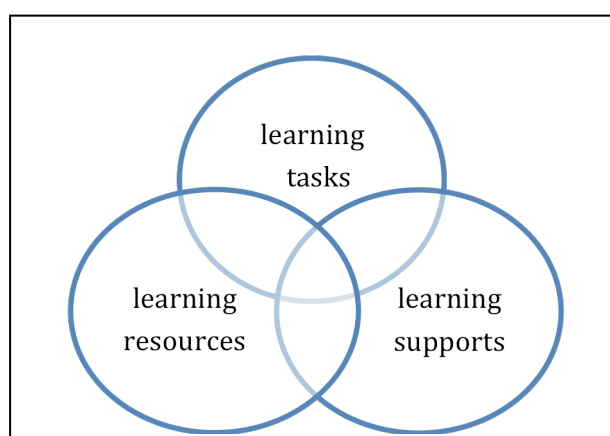


Figure 1. Element of learning design (REF) Oliver and Herrington (2001), Agostinho, S, et al 2002

To facilitate sharing of designs Oliver (1999, 2001) and Oliver and Herrington (2001) formalized the learning design sequence through the use of graphical notation (Figure 2): squares to symbolize tasks, triangles to symbolize resources, and circles to symbolize supports. Activities may differ for different learners; the activity sequence may be represented by parallel or concurrent activity components for that section of the sequence. Similarly, resources and supports can be specific to one task or may be available for entire duration of learning experience (Agostinho et al, 2008). These authors emphasise the significance of planning specific roles for students, the teacher and the technology in the learning environment (Oliver, 1999).

Koper & Olivie (2004) define learning design as the ‘application of a pedagogical model for a specific learning objective, target group and a specific context or knowledge domain’ (p. 98). It can encompass both the students’ and instructors’ activities and may involve the use of physical resources or the steps of the teaching and learning process.

Koper (2006) and Dalziel (2007) describe learning design as a simplification process, or educational modelling, by teachers of a sequence of learning activities within a unit of learning which support students in the classroom. In the e-learning environment learning design can provide scaffolding which supports students. Teachers can provide strategies, resources and links that the students are able to access to complete tasks and to develop their knowledge (Oliver, Herrington, & Omari 1996, p2).

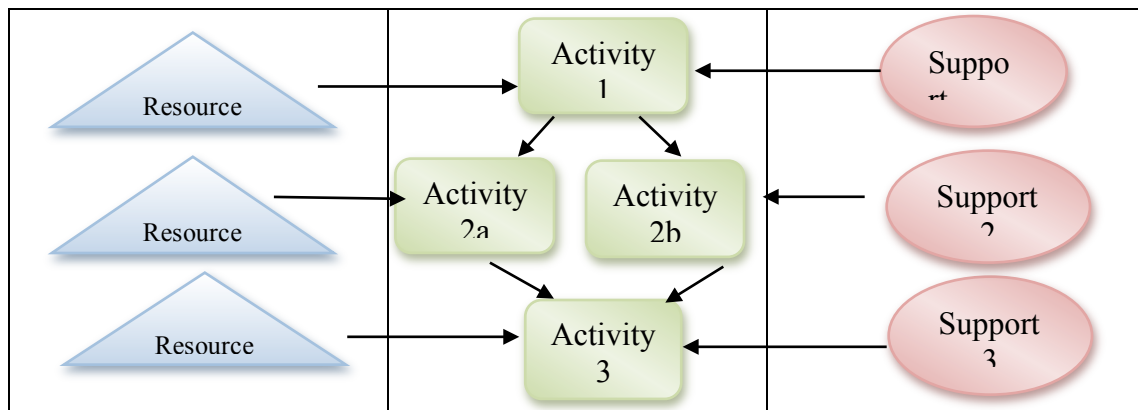


Figure 2. Example of concurrent activities in a learning design (LDVS) adopted from Agostinho et al, 2008.

Agostinho (2009), Oliver (2007) and Conole & Fill (2005) discuss representations of learning designs, which they describe as the outcome of the process of designing and planning a sequence of interactive learning activities. The creation of learning designs, representing the process of learning, leads to the possibility of them being shared, adapted and reused by other teachers. Building on others' designs might assist lecturers to design high quality learning environments. As such learning design is “a formalism for documenting teaching and learning practice to facilitate sharing and reuse by teachers” (Agostinho, et al 2009, p. 11). Agostinho, et al 2009 identified six approaches to representing learning designs: Educational Environment Modeling Language, computer readable format, the software application Learning Activity Management System, Learning Design Visual Sequence (LDVS), a lesson plan and Patterns. According to Agostinho (2006) and Agostinho et al (2009) a graphical representation, such as LDVS, can help teachers in understanding a learning design. The LDVS they illustrate uses maps to distinguish resources, tasks and supports in a subject.

Level of Learning Design

Masterman, Jameson & Walker (2009, p. 3) believe that learning design has three levels: as a technological infrastructure, as a framework for practice, and as a way to model and share practice through appropriate representations. Technological infrastructure is the development of an infrastructure of authoring applications to apply to a model of teaching and learning, such as the Learning Activity Management System (LAMS). In the second level the learning design is seen as a conceptual framework for practice. It includes designing, planning and orchestrating learning activities as part of a learning session or programme. The third level is sharing the outcomes of the first two levels. This means learning designs can involve technical approaches with the use of any technology or software and be integrated with the activities designed to support learning. Boyle (2010, p. 662) argues that learning design has four levels: course design, designing or planning sessions, designing activities and designing learning objects.

Issues in Learning Design: Academic Staff

The spread of ICT has resulted in university staff changing the way they teach. Examples of ICT use include internet based activities, blogs, LAMS and the use of digital media for presentation, interaction, and communication in teaching. Learning experiences are enhanced when ICT is used to build interactive learning environments for users to share and to collaborate (O'Sullivan and Samarawickrema, 2008). ICT can provide an environment with structure to guide the activities and roles of the learners and teachers (Oliver, 2007). The ability of ICT to provide communication channels between students allows them to be more active in collaborative learning. Combining ICT with learning design produces enhanced opportunities for sharing and re-using effective learning design. Waterhouse (2005) pointed out that e-learning can bring powerful changes in the way staff interact with students including the distribution of course information, the provision of web links, electronic communications, online testing and grading and how students interact with course content. In the mathematics education context, the requirement of effective online learning information and services can have a substantial benefit in enabling learners to access resources in variety of ways to decide which material to use and how to use it (Oliver 1999). However, conventional instructional design approaches tends to focus on online learning from the perspective of content delivery and produce learning settings whose main organising element is the course content (Oliver & Herrington, 2003).

There are a variety of issues and challenges facing teachers in using learning designs, particularly regarding the use of technology. Many university teachers have little expertise in the development of online learning environments. For example, the effective use of a learning design can help students to undertake complex activities by giving them an idea about the way to engage in the activity sequence. However, "teachers who expect students to work individually on online units are not only denying them the benefits of collaboration, but also the benefits of expert assistance providing hints, suggestions, critical questions, and the 'scaffolding' to enable them to solve more complex problems" Oliver (2001). To use learning design effectively requires staff to be familiar with new approaches to teaching. Many teachers are concerned with how to engage students and to have students active in learning processes: "educators find themselves challenged to plan engaging and effective learning experiences for students" (Agostinho, at el, 2009, p11).

Due to a combination of factors such as research, professional engagement and teaching commitments, academics often lack the chance to participate in workshops relating to the implementation of modern pedagogies in their educational setting (Agostinho, at el 2009). In addition, the skills and understandings of learning that many teachers develop through their face-to-face teaching are often insufficient to support their needs in online learning settings (Oliver, Herrington 2001). The challenge for lecturers is identifying how to provide an environment that can accommodate individual students' needs, promote deeper approaches to learning and engage students as active participants in learning experiences with use of ICT.

Benefits of learning design

One advantage of learning design is that it allows teachers to move away from a focus on content to better describe and share the teaching process (Dalziel, 2007, p1). The use of learning design can improve the student use of ICT by placing students into an environment where they can better relate them to context and practice (Oliver, 1999).

Students are more likely to use their knowledge and skills to connect new learning

with previous and related learning if they are able to share and discuss ideas in an interactive learning environment. Good learning designs engage learners in building on their expectations and provide students with the confidence to be critical of both themselves and their peers in a supportive environment (Oliver, 2007). Therefore, learning design involves identifying strategies for teachers to encourage students to share their thinking with others and reflect upon their learning. This encourages self awareness in the knowledge creation process (Oliver, et al, 2002), promoting creativity from students who are involved in the educational process. Although using technology in teaching can be effective, the key points to consider are: whether or not this tool adds value to education, how to engage students to use it and what strategies staff should use to create a balance between teaching and learning.

This Study

This aim of study is to examine the application of learning design principles to the redevelopment of an introductory mathematics subject. This study builds upon Oliver's (1999, 2001) and Herrington & Oliver's (2002) model which directs attention to resources, tasks and supports in designing the online learning environment. Learning design is defined as a structural model of the educational process which includes support of students in learning, understanding, and performing in their subject of study and to assist teachers in the design of their subject. The focus in this study is on the redesign of the subject home page and the use of learning design as an interaction map to reveal the structure of the educational process and the impact of communicating this to students. Through this students should be enabled to understand the learning and teaching process as they engage with their discipline. The interaction map maps resources, the sequence of student activities and learning support.

Re-design of the Home Page

This redesign involved changes to both the home page and the secondary level of pages. Both the original and redesigned E-Learning subject site contained tasks or learning activities, such as online quizzes and tutorial sheets; the learning resources, including solutions to mathematics questions covered in lectures, collections of worked examples and supports such as access to the E-Learning discussion forum. The structure of the original website did not convey the temporal sequencing of activities, including the appropriate time to access resources and to complete tasks (Refer Figure 3).

The need to organize resources in an e-learning environment to help students study effectively by providing good support and appropriate "scaffolding" has been emphasised by Xiaozhen & Yun (2002) and Hua Kuo (2008). Furthermore, the e-learning environment can provide "enhanced input and abundant learning resources and aids" Hua Kuo (2008, p. 297).

Oliver & Herrington (2003) argue there is a "need to plan learning settings based on meaningful and relevant activities and tasks which are supported in deliberate and proactive ways". There are many ways to improve teaching and learning by organizing the structure of e-learning sites and providing appropriate online learning strategies. The learning activities, learning resources and learning supports suggest an organizational framework for the e-learning setting (Oliver and Herrington, 2003).

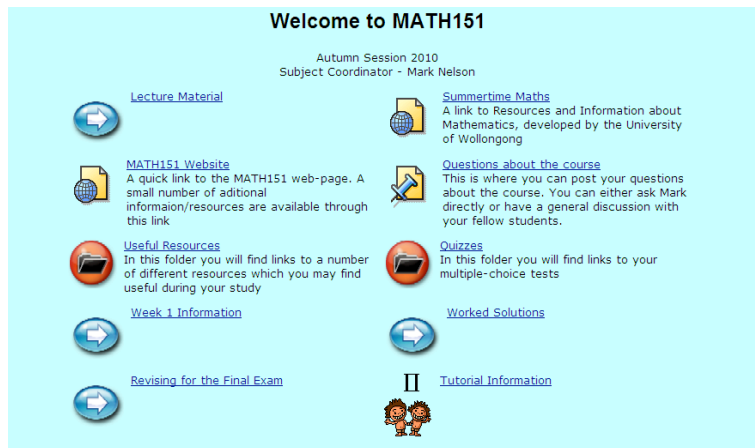


Figure 3. The original e-Learning website for math 151.

In the redesign, with a focus on resources, tasks and supports and the temporal requirements associated with their use, additional learning supports and resources were deemed desirable. The redesign at the home page level (compare Figure 3 and Figure 4) associated tasks, assessment, tutorial and projects with the resources or units to be learned and provided a link to general support. It also provided links to useful resources, past examination papers and solutions, and this has not changed in the initial re-design. However, the location of the introduction link was moved to the first position reflecting the temporal order of requiring information and it included a one page assessment diary.

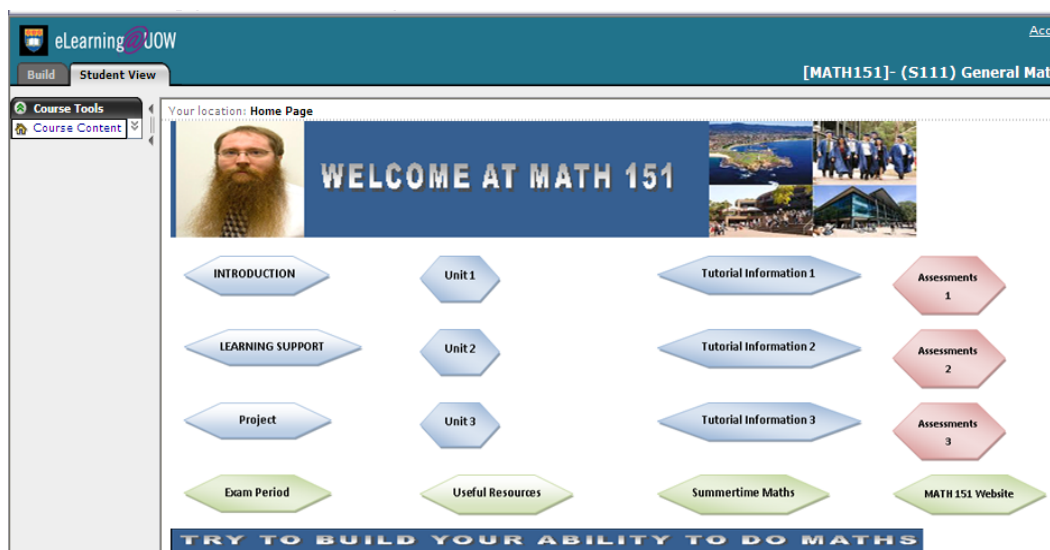


Figure 4. the redesign at the home page level.

Redesign/ Design of second level of pages

Introduction

In the original design the week 1 link provided students with access to subject information (subject outline, details of who is lecturing, assessment, details on tutorials and other policy information). In the redesign with an emphasis on student support a video presentation was added which provided an overview of the units and advice on how to effectively complete the subject. An assessment schedule providing a map of temporal aspects of the subject was also added.



Figure 5. Week 1 information and introduction.

Unit pages

In the original design “lecture material” referred to self-test and solutions, worked examples and solutions, practice online quizzes and solutions for all topics. There was much debate amongst the design team regarding the classification of tasks, resources and supports. In the first stage of redesign the topics were placed into three unit links which provided worked examples and solutions. This was primarily because there were three assessment tasks for each unit of work, although at this stage assessment was in a separate link. The remaining resources, categorised by topic, were moved to the assessment links associated with each of the three units. Additional resources were moved to the resources link. A major change is that each chapter (reading material and lecture material) previously given only in print and face-to-face format is now included in the units along with a video providing an overview of each chapter.

The next stage of the mapping process should be holistic, as suggested in the LDVS (Refer Figure 2). This suggests that the tutorials, assessments and additional resources, currently separate links should be included in the LDVS map. While the first redesign moved the e-learning page structure closer to the LDVS it has not associated each task or class of tasks to specific support and specific resources. To obtain such a map involves skills such as the use of image maps, icon creation, and PDF creation from Math type programs. The implementation of the map is also far more time-consuming than simply uploading the content. Perhaps this explains why the design of many subjects has remained content oriented. Casey and Dyson (2009, p.176) emphasize that “the implementation of any new pedagogical approach is time-consuming and highly labour intensive”. The inclusion of assessment into the map on the left to the homepage is a particular instance for debate: the lecturer wanting to promote the ease of being able to place changing assessment components into the website over the desire to include them in the holistic map. (Refer Figure 6)

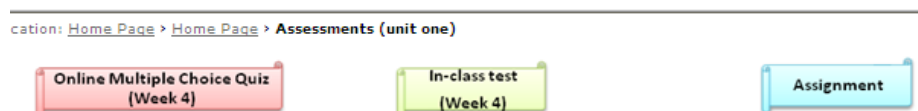


Figure 6. Assessment in the homepage.

Learning support

At the homepage level the previous link to the “questions about the course”, a discussion forum where students and lecturer provided feedback or answered questions, was expanded to become a link to Support. This linked to a more general list of support, including the discussion forum, including links to other sources of help such as a formal peer support program (PASS), the consultation hours of the subject lecturers and contact

details for a newly employed mathematics learning developer.

There is a second type of support that may be provided through resources. Video clips, self-tests, worked examples and solutions can be used to support the learning of specific mathematics skill required. In the desired LDVS design these would be associated with each task or class of tasks (Bukhatwa et al, 2011).

Resources added to the sites include orientation videos, videos of worked examples and web links; and, learning support, such as schedules, self tests. The structure proposed by implementing the LDVS in this work highlights to the designer when new supports or resources are required, for example in this subject there are video resources for seven topics but not yet for the three modelling, differentiation and integration.

The aim of this work has not been as an aid to the designer, rather the ultimate aim has been to communicate to students, the objectives of the chapters, the activities they must perform and the support that is available to help them to both complete the activities and to learn. "In this way the definition of learning design has been extended from guiding the lecturer in the design of learning through a learning design map to using a map to improve student's awareness of what they have to do, when it has to be completed and the resources available"(Bukhatowa et al, 2011). At this stage it would be possible to evaluate the impact of the LDVS on student learning.

Analysis of the Evaluation

MATH151 was the subject that was redesigned in this study. The students were drawn from different disciplines and were lectured by one of two lecturers; one taking the first seven weeks of the subject with the second lecturing the last six weeks. In addition students had small group tutorials of about 15 students for 1 hour per week. The evaluation was conducted in during the last lecture of the autumn session 2010 and after implementation of the redesigned eLearning pages in 2011.

The subject had 130 students enrolled this during the autumn term of the 2010 academic year 101 students responded to the survey. During the autumn term of the 2011 there were 146 students enrolled but with fewer students attending the final lecture there were only 49 students responded to the survey. "While a large percentage of students (79.3%) in 2010 indicated that E-Learning was useful in helping them learn and understand, it was clear from the ranking of different features of the E-Learning site such as the clarity, structure and student comments that the E-Learning could be better designed." (Bukhatwa et al 2011) and this was confirmed when a greater proportion of students found the pages better in a variety of ways (Refer Table 1).

Several significant differences were observed when comparing the value of design of e-learning in 2010 and 2011. These include: better; access material ($\chi^2=26.4$, $df=1$, $p\text{-value}=.000$); understanding ($\chi^2=28.4$, $df=1$, $p\text{-value}=.000$); identification of resources to support through difficulties in learning ($\chi^2=37.4$, $df=1$, $p\text{-value}=.000$); clarity as to what the lecturer wants ($\chi^2=36.9$, $df=1$, $p\text{-value}=.000$). Knowledge of the required assessment ($\chi^2=31.3$, $df=1$, $p\text{-value}=.000$).

The comments in 2010 suggested improvements could be made. Many students indicated that the e-learning was messy and difficult to find worked solutions to revise and further that they needed more examples that were better structured and accessible for revision (Bukhatwa et al, 2011, p. 4). However in 2011, students indicated that E-Learning "Very well organized easy to find things", "It is better customized", "It is easier to find what is needed", "Very different-clear", "Easy to see what part of course we are up to", "Very easy to navigate and user friendly".

Table 1: Useful of E-Learning Page

What did you gain from the design of e-learning E-learning pages structured to be helpful for students - How useful E-learning page in helping you understand in this subject	Agree & Strongly Agree	
	% 2010	% 2011
The E-learning page better organize work and better access learning material	79.3	83.6
The structured of E-learning pages are to help me understand objectives of the subject	76.3	83.6
The E-learning page helps me identify resources to support me through difficulties in my learning	74.3	79.5
The E-learning pages are structured in such a way that “what the lecturers wants from me is clear”	64.4	73.1
The E-learning pages are structured to help me know what is required in terms of assessment	75.3	79.6

The evaluation questionnaire also involved evaluation of the usefulness all resources identified by the lecturer and used in a subject (lecturers, notes, assessment...) but no differences were found between the two cohorts. Similarly no significant differences were found in perceived competency in mathematics topics.

Conclusions

Tertiary education has benefited from advances in technology, especially in the area of e-learning support for student learning. E-learning in mathematics is an area that requires further research in terms of applying different learning designs and pedagogical practices. The initial redesign has led to improvements in students' perceptions as to the functionality of the E-Learning site. While undertaking a website redesign many issues are illuminated for example, how to best show the interrelationship between resources, activities and support. From the perspective of team development: multiple people lead to multiple perspectives giving rise to questions as to whether or not materials should be mapped as resources or as a learning support. These issues complicate the design process. The learning design needs to take into account operational difficulties where for example to place frequently changing items such as assessment. Although there is a huge demand in terms of redesigning, it is anticipated that the benefit to students would out-way the costs and that after the first redesign subsequent changes would be less complicated, certainly it has allowed the design team to identify missing support and resources.

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The Use of R Language in the Teaching of Statistical Inference

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Statistical software packages like Minitab are commonly used in the teaching of statistics. The strength of these packages lies in the ease of performing standard statistical procedures such as hypothesis testing. However, such an ease can also be a hindrance to the understanding of the statistical principles underlying the procedures. Furthermore, these packages are not free and may not be cost-effective to implement. The R language is a powerful software for data analysis within which many statistical procedures have been implemented. The strength of R derives from its many capabilities besides being a data analysis tool. In this paper, we explore the simulation and graphing capabilities of R in teaching statistical inference, so as to enhance students' conceptual understanding of the inference methodology.

Keywords: R language; statistical inference; simulation

Introduction

The R language is a powerful software for data analysis within which many statistical procedures have been implemented. R was initially written by Robert Gentleman and Ross Ihaka of the Statistics Department of the University of Auckland, and can be considered as a “free” implementation of the S language, which was originally developed at Bell Laboratories. R is an official part of the Free Software Foundation's GNU Project, and is distributed as Free Software under the terms of the GNU General Public License. Precompiled R binaries as well as the R source code can be downloaded from <http://www.r-project.org/>.

As R is highly flexible and extensible, the use of R has not been limited to data analysis and statistical research. Coupled with its open-source nature and free availability, R is now commonly used in the teaching of statistics. For example, Dalgaard (2008) and Verzani (2005) considered using R for introductory statistics. Using a series of case studies and activities, Horton, Brown and Qian (2004) described how R can be used in a mathematical statistics course as a toolbox for exploration. In emphasizing that data analysis cannot be learned without actually doing it, Faraway (2005) has chosen to use R to teach linear models due to its “versatility, interactivity, freedom and popularity”. Exploiting the versatility of R as both a statistical software and a programming language, Cheang (2007) discussed how R can be used to enhance students' understanding of the principles in regression modelling, by providing insight into the “black box” which generates the regression output.

An attractive feature of R is its simulation capability. We can easily simulate observations from common distributions like normal and exponential. For example, the R command `rnorm(30, 0, 1)` generates a random sample of size $n = 30$ from $N(0, 1)$ distribution. In discussing the use of R in teaching financial mathematics and statistics, Stander and Eales (2011) showed how the simulation capability of R can be employed to illustrate features of the Monte Carlo methodology in estimating option price.

Another attractive feature of R is its graphing capability. For example, in teaching continuous bivariate distribution, the author has used R to draw perspective plots of a

joint density surface from different viewing angles. This approach has allowed students to “see” the probability (as volume under the surface) over a certain region of the x - y plane. R has been adopted by Keen (2010) in discussing graphics for statistics and data analysis “because its graphing capabilities are state-of-the-art”.

Using the flexible graphing capability of R, simulation results can easily be presented in publication-quality plots according to user’s need. In teaching the Central Limit Theorem (CLT), Cheang (2009) explored how the simulation and graphing capabilities of R can be used to provide students with better insight into the issues encountered. For example, how large should the sample size n be for CLT to hold? Is the rule of thumb $n \geq 30$ adequate? In this paper, we explore how these two capabilities of R can be employed to implement some teaching strategies for statistical inference. Based on the author’s experience, these strategies help to enhance students’ conceptual understanding of the inference methodology.

Use of R in Teaching Confidence Intervals

In statistical inference, one routine procedure is the construction of confidence intervals to estimate a population parameter. One way to interpret a confidence interval, say 90%, is that if we repeatedly draw samples from the population, then the calculated interval will contain the parameter for 90% of the samples drawn. We can impress this interpretation upon students through simulation of intervals. Also, students often (wrongly) think that the population parameter is a random quantity (because it is unknown), while the interval is fixed (because it is calculated). The simulation would help to eradicate such misconception.

Simulation of t -intervals for μ from normal population with unknown variance

Suppose we want to estimate the mean μ of a normal population with unknown variance based on a sample of size n . A $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{x} \pm t_{n-1}^{(\alpha/2)} \frac{s}{\sqrt{n}},$$

where $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ is the sample variance, and $t_{n-1}^{(\alpha/2)}$ denotes the $(1-\alpha/2)$ -quantile of the t_{n-1} -distribution.

Wild and Seber (2000, p. 332) presented simulated 95% t -intervals for μ from a normal population. The R code in Appendix A.1 shows how such a simulation can be performed using R. Figure 1 illustrates simulated 90% t -intervals for μ from normal population with $\mu = 50$, when the sample size is $n = 10$. For the 20 samples shown, we see that the intervals based on two of the samples (indicated by “out”) do not contain μ . This is indeed consistent with the interpretation of 90% confidence intervals. In addition, by showing the steps that calculate these intervals, the R code will help students understand the “black box” which generates them in a typical inference output.

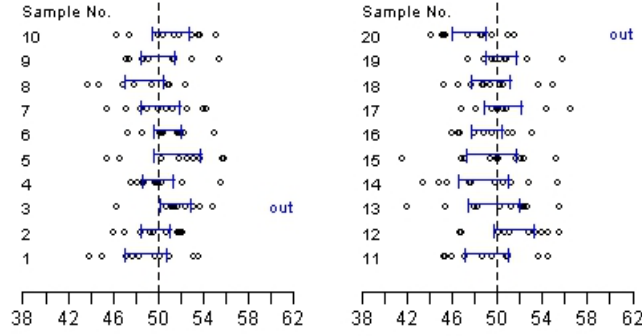


Figure 1. Simulation: 90% t -intervals for μ from normal population with unknown variance ($\mu = 50$, $n = 10$).

Simulation of t -intervals for $\mu_1 - \mu_2$ from normal populations with equal variances

Suppose we want to compare the means μ_1 and μ_2 from two normal populations with equal variances based on independent samples of sizes n_1 and n_2 . A $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$\bar{x}_1 - \bar{x}_2 \pm t_{n_1+n_2-2}^{(\alpha/2)} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where $s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$ is the pooled variance.

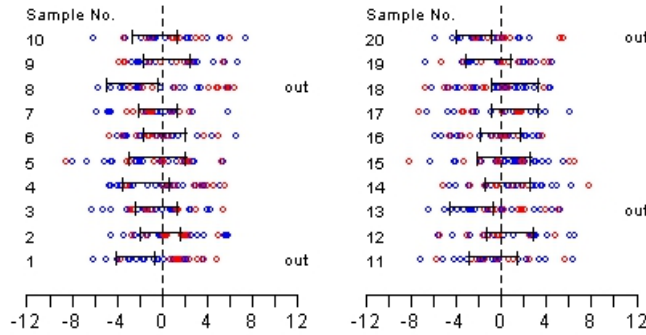


Figure 2. Simulation: 90% t -intervals for $\mu_1 - \mu_2$ from normal populations with equal variances ($\mu_1 = \mu_2 = 0$, $n_1 = 20$, $n_2 = 10$).

Figure 2 displays simulated 90% t -intervals for $\mu_1 - \mu_2$ from normal populations with equal variances, when the sample sizes are $n_1 = 20$ and $n_2 = 10$. The R code to perform this simulation is given in Appendix A.2. Since one usual purpose of constructing an interval for $\mu_1 - \mu_2$ is to determine whether μ_1 and μ_2 are significantly different, we take $\mu_1 = \mu_2$ in simulating these intervals. To present both the data points and the intervals on the same plot, we further take $\mu_1 = \mu_2 = 0$. For the 20 replications shown, we see that the intervals based on four of the replications (indicated by “out”) do not contain the difference $\mu_1 - \mu_2 = 0$. We can verify through simulation of further samples that indeed approximately 10% of the intervals would not contain 0.

Simulation of χ^2 -statistic for sample variance

For a normal population with variance σ^2 , we know that the sample variance S^2 has a χ^2 -distribution given by

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2.$$

The $100(1-\alpha)\%$ confidence interval for σ^2 constructed based on this distribution is

$$\left[\frac{(n-1)s^2}{\chi_{n-1}^{2(\alpha/2)}}, \frac{(n-1)s^2}{\chi_{n-1}^{2(1-\alpha/2)}} \right],$$

where $\chi_{n-1}^{2(\alpha/2)}$ denotes the $(1-\alpha/2)$ -quantile of the χ_{n-1}^2 -distribution. Notice that this interval does not have the usual form of

parameter estimate \pm critical value \times st. dev. of estimate.

The unusual form of the interval is a direct consequence of the unusual form of the statistic $V = (n-1)S^2/\sigma^2$ from which the interval is derived. To impress this “anomaly” upon students, one approach is to convince students that such unusual form is needed to give a χ^2 -distribution. The R code in Appendix A.3 shows how the distribution of V can be simulated. In Figure 3(a), the density histogram displays the empirical distribution of V for 10000 samples of size $n = 30$ drawn from a normal population. The χ_{n-1}^2 p.d.f. superimposed indicates that V indeed has a χ^2 -distribution. The normal p.d.f. (with the same mean and variance as that of χ_{n-1}^2) superimposed would help to eradicate the common misconception that V has a normal distribution.

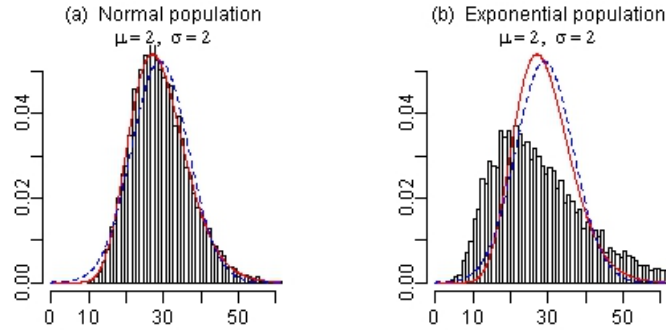


Figure 3. Simulation: Empirical distribution of $V = \frac{(n-1)S^2}{\sigma^2}$ for normal and exponential populations ($n = 30$), with χ_{n-1}^2 (solid curve) and $N(n-1, 2(n-1))$ (dotted curve) p.d.f.'s superimposed.

Furthermore, when learning the interval for σ^2 , students often (wrongly) think that the normality condition is not necessary when n is large (due to their misuse of Central Limit Theorem). Figure 3(b) shows the density histogram obtained for $n = 30$ when the population is exponential. The χ_{n-1}^2 p.d.f. superimposed does not match the empirical distribution of V , thus indicating the necessity of a normal population in constructing confidence interval for σ^2 , even when n is large.

Use of R in Teaching Hypothesis Testing

In statistical inference, the procedures of hypothesis testing and confidence interval estimation are closely related. For example, in testing the null hypothesis $H_0: \mu = \mu_0$ against a two-sided alternative $H_1: \mu \neq \mu_0$, H_0 is rejected at the α level of significance if and only if μ_0 falls outside the $100(1 - \alpha)\%$ confidence interval for μ .

The R code in Appendix A.1 can easily be modified to include calculation of the one-sample t -statistic,

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}.$$

For 90% t -intervals in Figure 1 that do not contain $\mu_0 = 50$, we can verify that $|T| > t_{n-1}^{(0.05)}$, where $n = 10$ is the sample size. Furthermore, we can demonstrate to students that approximately 10% of the samples generated from a normal population with $\mu = 50$ would actually have $H_0: \mu = 50$ rejected. This should enhance their understanding of the meaning of α .

Simulation of Welch's t -statistic and two-sample t -statistic

When testing $H_0: \mu_1 = \mu_2$ for two normal populations and equality of variances cannot be assumed, one statistic that can be used is the Welch's t -statistic T_1 given by

$$T_1 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}.$$

T_1 has a t -distribution under H_0 , where

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\frac{1}{n_1 - 1} \left(\frac{s_1^2}{n_1} \right)^2 + \frac{1}{n_2 - 1} \left(\frac{s_2^2}{n_2} \right)^2}.$$

The two-sample t -statistic T_2 requires the two population variances σ_1^2 and σ_2^2 to be equal, so that

$$T_2 = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1 + n_2 - 2} \text{ under } H_0.$$

As a computer assignment, we can use the simulation capability of R to study the empirical levels of T_1 and T_2 . To assess the impact of assuming equal variances on the empirical levels, we consider different ratios of $k = \sigma_1/\sigma_2$, including $k = 1$. The R code to perform this simulation is given in Appendix A.4. Table 1 gives the empirical levels of nominal 5%-level t -test of $H_0: \mu_1 = \mu_2$ versus $H_1: \mu_1 \neq \mu_2$ when $n_1 = 20$ and $n_2 = 10$ (10000 replications), that is, the proportions of replications in which

$$|T_1| > t_v^{(0.025)} \text{ and } |T_2| > t_{n_1 + n_2 - 2}^{(0.025)}.$$

These simulation results seem to suggest that for $n_1 > n_2$, the empirical level of T_2 is always less than that of T_1 . Can we then conclude T_2 is “better” than T_1 even when the variances are unequal, in the sense that T_2 has a smaller probability of Type I error? What if $n_1 < n_2$ or $n_1 = n_2$? How about the probability of Type II error? With R “freely” available, such exploratory questions can be further investigated by students in project work.

Table 1. Empirical levels of nominal 5%-level t -test of $H_0: \mu_1 = \mu_2$ versus $H_1: \mu_1 \neq \mu_2$ ($\mu_1 = \mu_2 = 0, n_1 = 20, n_2 = 10$), based on 10000 replications.

$k = \sigma_1/\sigma_2$	Welch's t -statistic T_1	Two-sample t -statistic T_2
1	0.0486	0.0461
2	0.0492	0.0206
5	0.0519	0.0101

Conclusion

By exploiting its simulation and other capabilities, R has the potential to be an effective teaching and learning tool for statistics. In a sense, R is an “evolving” language as every user is also a developer. With its ease of adaptability according to user’s need, the R language provides a platform for statistics educators to “freely” explore how technology can be integrated into their teaching to enhance students’ understanding.

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Appendix: R Codes

In this appendix, we give the R codes for some of the examples discussed. A good introduction to the R language is *The R Manual* edited by the R Development Core Team and downloadable from <http://www.r-project.org/>.

A.1: Simulation of t -intervals for μ with unknown variance

```
set.seed(238)
alpha <- 0.10
n <- 10
mu <- 50
sigma <- 3

# t-interval
cvalue <- qt(1-alpha/2,df=n-1)

nr <- 2 # No. of plots
r <- 10 # No. of samples per plot

for (j in 1:nr)
```

```

{ x <- matrix(NA,r,n)

for (i in 1:r)
{ x[i,] <- rnorm(n,mu,sigma)
}

plot(0,0,type="n",bty="n",axes=F,xlab="",ylab="",mgp=c(2,0.5,0),
xlim=mu+c(-1,1)*4*sigma,ylim=c(0,r+1),cex=0.7)
axis(side=1,at=seq(mu-4*sigma,mu+4*sigma,2),mgp=c(2,0.5,0),cex=0.7)
abline(v=mu,lty=2)

for (i in 1:r)
{ points(x[i,],rep(i,n),pch=21,cex=0.7)
text(mu-4*sigma,i,as.character(10*(j-1)+i),adj=0,cex=0.8)
xbar <- mean(x[i,])
s <- sd(x[i,])

# sigma unknown, n small
lower <- xbar - cvalue*s/sqrt(n)
upper <- xbar + cvalue*s/sqrt(n)

segments(lower,i+0.15,upper,i+0.15,lty=1,col="blue")
points(c(lower,upper),rep(i+0.15,2),pch="|",col="blue",cex=0.8)
if (lower>mu | upper<mu)
text(mu+4*sigma,i+0.1,"out",col="blue",adj=1,cex=0.8)
}
text(mu-4*sigma,r+1,"Sample No.",adj=0,cex=0.8)
}

```

A.2: Simulation of t-intervals for $\mu_1 - \mu_2$ with equal variances

```

set.seed(238)
alpha <- 0.10
n1 <- 20
n2 <- 10
mu1 <- 0
mu2 <- 0
sigma <- 3 # sigma1 = sigma2 = sigma

# t-interval
cvalue <- qt(1-alpha/2,n1+n2-2)

nr <- 2 # No. of plots
r <- 10 # No. of replications per plot

for (j in 1:nr)
{ x <- matrix(NA,r,n1)
  y <- matrix(NA,r,n2)

  for (i in 1:r)
  { x[i,] <- rnorm(n1,mu1,sigma)
    y[i,] <- rnorm(n2,mu2,sigma)
  }

  plot(0,0,type="n",bty="n",axes=F,xlab="",ylab="",mgp=c(2,0.5,0),
xlim=4*sigma*c(-1,1),ylim=c(0,r+1),cex=0.7)
axis(side=1,at=seq(-4*sigma,4*sigma,2),mgp=c(2,0.5,0),cex=0.7)
abline(v=0,lty=2)

for (i in 1:r)

```

```

{ points(x[i,],rep(i,n1),pch=21,col="blue",cex=0.7)
  points(y[i,],rep(i,n2),pch=21,col="red",cex=0.7)
  text(-4*sigma,i,as.character(10*(j-1)+i),adj=0,cex=0.8)

  xbar <- mean(x[i,])
  ybar <- mean(y[i,])
  s1 <- sd(x[i,])
  s2 <- sd(y[i,])
  sp <- sqrt(((n1-1)*(s1^2) + (n2-1)*(s2^2))/(n1+n2-2))

  lower <- (xbar - ybar) - cvalue*sp*sqrt(1/n1 + 1/n2)
  upper <- (xbar - ybar) + cvalue*sp*sqrt(1/n1 + 1/n2)

  segments(lower,i+0.15,upper,i+0.15,lty=1,col="black")
  points(c(lower,upper),rep(i+0.15,2),pch="|",col="black",cex=0.8)
  if (lower>0 | upper<0)
    text(4*sigma,i+0.1,"out",col="black",cex=0.8)
}
text(-4*sigma,r+1,"Sample No.",adj=0,cex=0.8)
}

```

A.3: Simulation of empirical distribution of sample variance

```

set.seed(238)
n <- 30
mu <- 2 # mu = sigma for exponential
sigma <- 2

r <- 10000 # No. of samples
Vstat <- rep(NA,r)

for (i in 1:r)
{ x <- rnorm(n,mu,sigma) # Normal population
  # x <- rexp(n,rate=1/mu) # Exponential population
  s2 <- var(x)

  # V-statistic
  Vstat[i] <- (n - 1)*s2/sigma^2
}

upper <- 8*sqrt(2*(n-1))
chisqpdf <- dchisq(seq(0,upper,0.1),n-1)
Npdf <- dnorm(seq(0,upper,0.1),n-1,sqrt(2*(n-1)))

# Plot density histogram of V-statistic
hist(Vstat,breaks=seq(min(Vstat),max(Vstat)+1,1),prob=T,right=T,
main="",xlim=c(0,upper),ylim=c(0,max(chisqpdf)),mgp=c(2,0.5,0),cex=0.7
)

# Plot chisq-pdf
lines(seq(0,upper,0.1),chisqpdf,lty=1,col="red")

# Plot normal pdf
lines(seq(0,upper,0.1),Npdf,lty=2,col="blue")

mtext(side=3,line=1,outer=F,"(a) Normal population",cex=1.0)
# mtext(side=3,line=1,outer=F,"(b) Exponential population",cex=1.0)
mtext(side=3,line=0,outer=F,paste("m = ",mu,"", s =
", sigma, sep=""), font=5,cex=1.0)

```


A.4: Simulation of empirical levels of Welch's t-statistic and two-sample t-statistic for testing $\mu_1 = \mu_2$

```
set.seed(238)
alpha <- 0.05
n1 <- 20
n2 <- 10
mu1 <- 0
mu2 <- 0
k <- 2
sigma2 <- 1
sigma1 <- k*sigma2

r <- 10000 # No. of replications
tstat <- matrix(NA,r,2)

cvalue <- rep(NA,2)
cvalue[2] <- qt(1-alpha/2,n1+n2-2)

# Count no. of times H0: mu1 = mu2 (vs H1: mu1 neq mu2) is rejected
count <- rep(0,2)
names(count) <- c("Welch's t","Two-sample t")

for (i in 1:r)
{ x <- rnorm(n1,mu1,sigma1)
  y <- rnorm(n2,mu2,sigma2)
  xbar <- mean(x)
  ybar <- mean(y)
  s1 <- sd(x)
  s2 <- sd(y)
  sp <- sqrt(((n1-1)*(s1^2) + (n2-1)*(s2^2))/(n1+n2-2))

  # Welch's t-statistic
  tstat[i,1] <- (xbar - ybar)/sqrt(s1^2/n1 + s2^2/n2)

  # Two-sample t-statistic
  tstat[i,2] <- (xbar - ybar)/(sp*sqrt(1/n1 + 1/n2))

  a <- s1^2/n1
  b <- s2^2/n2
  df <- ((a + b)^2)/(a^2/(n1-1) + b^2/(n2-1))
  nu <- floor(df)
  cvalue[1] <- qt(1-alpha/2,nu)

  if (abs(tstat[i,1]) > cvalue[1])
  { count[1] <- count[1] + 1
  }
  if (abs(tstat[i,2]) > cvalue[2])
  { count[2] <- count[2] + 1
  }
}
level <- count/r
print(level)
```

Out of the Ashes – A Case Study

Christchurch New Zealand Post Earthquake 22/02/2011. A Paradigm Shift, Students and Staff Have a Radically Altered Environment. How to Respond?



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Introduction

Following a second significant Christchurch earthquake in six months, on 22 February 2011, we had a challenge. The 6.3 earthquake happened at 1:51pm, just after morning lectures were completed on the second day of the first semester at the University of Canterbury (UC). Many classes had not yet started. Introductory Statistics, STAT101, was one of these courses. Course outlines had been printed and lecture notes for lecture 1 were ready to be distributed to the 400+ enrolled on the course for the first lecture on Wednesday 23rd February. The lecture did not take place! We (staff and students) were sent home to deal with personal issues. Getting home took more than 2 hours, a journey that usually takes 15 – 20 minutes depending on the time of day. Then followed days and, in some cases, weeks, of dealing with basic life functions; with no power, no water and no sewerage. Community work helped pass the time for many, with volunteers bringing some relief to those in the worst affected areas. Very soon afterwards we were challenged by the university recovery team to move forward and make provisions for offering papers using on-line and flexible formats. To continue, UC had to get up and running and we were needed to help in the recovery. A not inconsiderable task for many, with the stress related to entering buildings, but a real challenge to those with the vision to look beyond the ‘normal’ and find ways to bring education to those who sought it in an earthquake-stricken city. Working from the dining room table, with a view over the city, a team of motivated and innovative lecturers and tutors and some embryonic ideas for alternative formats, we responded to the challenge. The university post-earthquake comprised primarily of a city of large tents, to be used as lecture theatres and tutorial spaces, with very few of the original buildings in a useable condition when the university reopened for business just three weeks after the earthquake closed its doors. With few social spaces on campus, one of

the tents was designated an aptly named café, 'Intencity 6.3', which became a popular informal meeting space given the limited bookable rooms available. However, audio-visual facilities were limited to whiteboards and microphones, so moving to an on-line environment seemed a better choice for the introductory statistics course. We had no idea how the students would respond to the altered environment, in terms of the physical campus as well as the on-line course. There were some real surprises in the student responses to the on-line course and many lessons learned, best of which? Don't make any assumptions, Statistical or otherwise.

We had a few logistical concerns including; do all the students have internet, computers at home or even power? Is there a computer lab available for face-to-face help for those students who need it? If so, can I book it? And get the students timetabled for regular sessions? The next step was to bring all the aspects together.

On-line learning environment

A main aim of STAT101 is to offer as many learning pathways as possible to enable all styles of learning to take something from the course. Having introduced changes in the introductory statistics course (Brown and David 2010, David and Brown 2010) we were able to introduce the concept of statistical literacy (Rumsey 2002, Wild and Pfannkuch 1999) and follow the leaders of the reform in the approach to the delivery of introductory statistics (Cobb, 1992, Garfield, 1994 and 1995, Moore, 1997 and 2005).

A large component of the course relies on online learning using Excel as a computational tool and for assessment purposes, but lectures are still an important method of delivery of the course content and the first learning opportunity for students. The challenge of the earthquake was in moving entirely to on-line delivery, even if for a limited (but unknown) period, and in a very short timeframe.

Running a course entirely on-line requires a change in delivery style and material – it was a huge learning curve to respond so quickly to a request to do so. There were already certain elements in place within STAT101 that enabled it to be up and running ready for the new start date of March 14th.

We had already introduced partially populated lecture notes (PPLN) as a means of delivering the course material lectures via Powerpoint. This style of lecture notes provides an effective learning medium that has been researched extensively (Tonkes, 2009). We drew on evidence to ensure best practice for the main means of delivery of course content, as it is understood that activating content by means of writing may significantly enhance students' comprehension of domain knowledge and promote their critical thinking skills (Tynjala, 1998). The state of the art lecture theatre equipment at UC allowed the completion of the notes within each lecture by means of electronic pens on computer screen. This technology also improves the student experience as the projection of the material onto the lecture theatre screen is clear without the distortion of OHP or hand writing on white board, especially important within the large lecture theatres required to cater for the numbers enrolled. Of course, post-earthquake this technology was no longer available when the university reopened!

Tonkes (2009) writes "Using PPLN students retain the opportunity to write during lectures, but at the end of the course there is a single set of lecture notes accessible to all on the class webpage. Of course, there is the danger that students may not take advantage of the chance to write during lectures, and may instead simply download the complete set of notes once it becomes available; but results indicate that this behaviour is uncommon. A high proportion of first-year students are unskilled in discerning the crucial parts of a presentation." As Weimer (1993) discovered upon inspecting student notes; 'The student notes and my notes bore little resemblance to each other'. A major

study (Harvey et al, 2006) of the first-year experience in the UK confirmed that first-year students have difficulty in developing effective study skills, and reports that any habits that can be instilled provide significant benefits through the remainder of their studies.

Our first response to the earthquake was to make lectures, that had been recorded in the previous year, available to students by burning them to DVD, as well as posting the videos on-line, in 'Learn', the Moodle based learning management system (LMS) in use at UC. This allowed students to view the lectures remotely, with pdf copies of the (partially populated) lecture notes available to view or print.

The structure of the course pre-earthquake had incorporated a range of alternate teaching formats and learning materials. Interactive weekly examples classes followed the week's lecture material. Using 'clickers' in these sessions encourage participation and provide the opportunity quickly assess whether important concepts have been understood.. Short examples clips using lecture capture with voice-over, duplicating the weekly Examples class examples, were already in production, and posted on Youtube <http://www.youtube.com/user/UCMathStat> The original intention was to allow students to use these if they prefer, rather than to attend the face-to-face examples classes. These YouTube clips were added to the on-line materials offered, and new material was recorded. When all the examples classes were recorded they were listed in 'Playlists' linked to the LMS. The exercises used in the example classes, further expanded with full working were also posted in pdf format on the course web site.

On-line tutorial questions had been used in the course since 2008. We use the quiz facilities in the LMS to provide these formative and summative learning opportunities. Over each occurrence of the course in subsequent years these questions and quizzes had been further developed and refined, and now formed the main method of monitoring student participation and motivation.

We were given, on three days each week, face to face tutorial sessions in a large (38 computers) computer lab, the only bookable computer lab on campus at that time. The need for this type of facility had been stated at the outset as a requirement for the course to run on-line as we firmly believed some students would need this space to work and would welcome the opportunity to meet and ask questions. This provision enabled us to extend the computer facility to all STAT101 students, enabling those without power or internet at home to make use of the lab with help and support on hand, if needed. Many students continued to work at home using the on-line resources. This allowed those who needed the facility to occupy the lab. We rarely had to turn away a student looking for a place to work.

The students had missed the first lecture of the course which is the general introductory welcome style address to the students. Most students are in their first year at University and being a very large class, it can be rather daunting. This first lecture has been essential to set the tone for the rest of the course. Without the opportunity to speak to the students the course web page had to offer this level of support. The website was the "face" of the course and it had to be presented in a way so that the students could easily read and understand how the course would run, and what was expected of them. The front page of the website, the first page viewed by a student logging in to the course, needed to be kept up-to-date. We made sure it was informative but not overly verbose, and any changes and up-dates were immediately obvious.

We already had a very clear course structure with carefully considered learning objectives (Brown and David 2010). STAT101 is normally presented in a weekly (12 weeks for the semester) format. In the LMS, links to each week's material were presented in a standard format. The learning objectives were stated first and the links to

the material ordered in a logical manner with explanatory labels. This investment in developing such a formatted structure in the course paid off when we had to move to web based delivery. At this point, in our post-earthquake course, the weekly on-line assessment and its deadline were placed in the penultimate position with the Forum, the place for questions and clarifications of the week's material, clearly labelled as the conclusion. The course is usually assessed by secure on-line skills tests, written assignments and a written final exam based on statistical thinking (David and Brown 2010). The plans for skills tests had to go 'on-hold' until physical spaces became available.

Post earthquake the opportunity for student – lecturer interactions was very much reduced. In response we actively encouraged the use of 'LMS' Forum posts. This was a valuable resource for on-line learners, enabling questions to be asked and answered and discussions to take place. More than 200 posts were made to the forums over the semester, each of which generated an average of 3 replies, many from students. From reading the posts, it was clear that there was usually an improvement in learning both for the students who originally posted the questions, and for the others who were able to read and monitor the discussions. Students were encouraged to ask and answer questions with tutor intervention as encouragement and confirmation. This timely feedback is understood to be of benefit to student learning and self –esteem (Nicol, 2006).

Another method of interaction was electronic feedback to students on the tutorials they submitted online. Generating this feedback, an exercise which continued through the semester, required an immense investment in time and effort, with an average of 20 to 30 aftershocks being recorded each day. By reviewing their submissions, students were able to produce improved work by resubmitting their assignments. The opportunity to resubmit their work is an important aspect of giving feedback to a student, and is the only way to tell if the feedback has had a positive result (Sadler, 1989). Interestingly, this is one of the most often forgotten aspects of formative assessment. Unless students are able to use the feedback to produce improved work, through for example, re-doing the same assignment, neither they nor those giving the feedback will know that it has been effective. (Boud, 2000, p. 158) The formative feedback added in to the on-line tutorial questions enabled students to correct their errors and improve their learning, and also offered low-stakes assessments (Nicol, 2006). Each tutorial was worth up to 1% of assessment when completed by the weekly deadline. The opportunity to submit, receive feedback and resubmit allowing students to self regulate and improve their own learning.

Discussion

It was not without trepidation that we began to offer this course through an on-line environment. STAT101 has a broad range of students from wide ranging backgrounds, both discipline and facility in the subject, bringing varying needs and levels of interest. Our philosophy was to consider all suggestions for improvement of the delivery of the course material and any new ways of engaging with the students. Students and staff were encouraged to use the forums to ask and answer questions, to share any on-line materials, found elsewhere on the internet with the class, and to work cooperatively to improve the learning experience of all. In a course which usually has a good deal of face to face interaction we were hoping to find alternative ways to deliver and assess the learning outcomes. Initially, in order to persuade as many students as possible to come in to the campus computer labs, we had lecture notes printed to hand out to students attending. In addition we provided regular up-dates, by email and forum postings, to

keep students current and make it clear what was required in terms of participation, in particular the low stakes (1%) weekly tutorial deadlines (Nicol, 2006). The LMS allowed us to track the levels of participation and to follow up on those students not engaging with the course. Our main concern was whether we would be able to assess the students in the normal secure skills tests since computer lab space was at a premium. In term 1 the university timetable changed weekly, as teaching spaces became available. As the term progressed we rarely knew what problems we would have to resolve on a daily basis, as the aftershocks continued to cause disruption, concern for safety was paramount. There was a constant feeling of uncertainty that we likened to living in a war zone.

As (AV enabled) physical lecture spaces gradually became available we decided to return to offering normal face to face lectures and examples classes in the second term, giving the students advance notification that this was the plan. The second term is always the time when the material becomes more demanding as we progress from descriptive statistics and probability to confidence intervals and inference. We continued to offer the same on-line environment and support for those students happy to continue working in this way. The students' response was excellent. The lectures and examples classes were popular as many students welcomed the return to real time class participation. In addition our own department computer lab spaces became available again, allowing the supervised skills tests to run, although in a different form to pre-earthquake. This enabled the summative assessment of basic skills, computations, methods and understanding. The tutorial sessions continued to attract those wanting and needing to seek support and we introduced drop-in help sessions on the days the tutorials did not run, to help anybody who needed to catch up. Although some students continued to work remotely from home, many welcomed the return to campus interaction, and the support of peers and tutors. For these reasons we are not planning to move to a completely on-line course.

Conclusion and Student Feedback

The conclusion of the semester was not without further disruption, with two severe aftershocks registering 5.3 and 6 respectively on June 15th closing the university again for several days causing severe disruption to the plans for study week and the examinations time table. At this point we decided it was time to join the students' Facebook group, set up by the class representatives, and try to offer reassurance and support. We made a request to join and became the 100th member of the group, 27% of the class were group members, though many other admitted to 'lurking' (watching the posts without contributing). Facebook interaction up to this point had been a social space without any tutors involved. The 'live chat' facility was a definite improvement on what was available through the LMS and allowed students on-line to ask questions and get immediate replies. This is one aspect of the on-line environment we intend to investigate further. The most important message we could offer was that the final exam would go ahead as planned and students should continue to prepare for it. By this time we believed most participating students wanted completion and the attendance for the final exam was at least as high as in previous semesters. The good news is that more A+ grades were won than in previous occurrences, possibly due to the on-line weekly participation keeping students on task, possibly those who continued on the course (post-quake) were more responsive. There were (not surprisingly) a number of students who left the city. This will provide important material for further investigation as the student cohort progress in their undergraduate study.

Feedback was sought after the completion of the course and results had been

released. We did not expect much response as the vast majority of the student cohort was underway with semester 2 studies, but were pleasantly surprised. 65 of the 358 students completed the survey and the some of comments have led to further adjustments in the course. Clearly the respondents were a self-selected sample and not all of the respondents made comments, but in response to the question:

“Are there any further comments you would like to make about the experience of STAT101 following the earthquake?” several stood out:

STAT101 was run really well after the earthquake. To have records of the lectures to watch from home was really great, and it was beneficial to be able to pause and rewind, or fast forward through material already done at school.

The online tutorials were extremely useful for me. I did them all numerous times to prepare for the exam.

I think the videos were very very good!!!! The ability to pause the lecture and go back to something i didnt understand was GREAT. The online Learn forum was very good as well. Those two (and the lecture notes online) I think are what helped me the most in doing well. The helpers in the computer sessions were also very useful.

While there was no lack on Online content, I would have appreciated more Online interaction, that of which we got in Tutorials/Examples Classes. Perhaps have a live chat at certain times of the day with tutors at the ready to help.

Being able to use the tutorial questions to practice even after the tutorials were closed was very useful. Also having practice tests for the skills tests available was probably the best way I was able to learn.

The interactive example classes with clickers were awesome, was a perfect way to finish off the week's lectures of stats (conveniently scheduled after all stat lectures for the week, on fridays) and understand the work better.

77% of the respondents were positive about the return to real-time lectures, 99% of them found the on-line tutorial questions either useful or very useful. Interestingly, none of the students rated STAT101 as worse than their other courses when asked: “How did the learning resources for this course compare with other courses you studied in 2011 semester 1?”

Table 1. “How did the learning resources for this course compare with other courses you studied in 2011 semester 1?”

Very much better	Somewhat better	About the same	Somewhat worse	Very much worse
18	29	17	0	0
0.277	0.446	0.262	0	0

No students selected ‘Somewhat worse’ or ‘Very much worse’.

From these responses we were satisfied with our response to the earthquake challenge and have determined that each successive occurrence of the course will include some reflection and time spent enhancing the learning environment.

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Using Technology to Provide a Supportive Mathematical Pathway into University

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As a result of Government initiatives implemented to broaden student engagement in learning, universities are striving to increase their mature-aged and low socio-economic numbers. As these students are often the most susceptible to attrition, providing the necessary guidance and support is essential. With many students from this demographic studying externally it is becoming increasingly important to innovatively engage them. Through the use of Tablet PCs, comprehensive Electronic Study Guides have been created that are able to effectively guide and support students completing bridging mathematics courses.

Keywords: electronic study guide; Tablet PC; external study, bridging mathematics, feedback, technology

Introduction

The changing social and economic climate has resulted in the Australian Government developing a vision of a socially inclusive environment [1]. This vision emphasises the importance of higher education and has resulted in a number of monetary incentives aimed at increasing the education participation rate of mature-aged students and students from low socio-economic (LSES) backgrounds [2]. As a result, a broader demographic of students is wishing to gain entry into University and with the current resource boom, many are attempting to enter engineering or science degrees.

CQUniversity Australia (CQU) has been consistently recognised as having the highest LSES participation rates of any Australian university (approximately 43%) and was awarded five stars for indigenous participation and cultural diversity of the student body [3]. In a recent 2011 'I'm all ears' Student Forum the CQU Vice Chancellor, Scott Bowman, acknowledged that opening the doors to more students must be accompanied by appropriate academic support [4].

To assist the transition of this 'non-traditional' demographic into university, CQU provides: a suite of bridging programmes (CQUniACCESS) to prepare students for entry into university (regardless of their former skills and educational achievements); dedicated academic support centres to assist undergraduate students with writing and mathematics; and in 2011 the university is trialling a new 'BE-Successful' readiness programme to identify and assist 'at risk' students prior to the commencement of their undergraduate programme.

The Mathematics Learning Centre (MLC)

The MLC's key responsibilities are to:

1. Deliver mathematics courses into CQUniACCESS programmes;
1. Provide academic assistance and resources to undergraduate students experiencing difficulty with a mathematical component of their programme.
2. Provide resources for students captured by the 'BE-Successful' readiness

programme.

Bridging Mathematics Courses

The MLC offers three levels of bridging mathematics courses into CQUniACCESS programmes:

Transition Mathematics 1 (TM1)

TM1 is a course in elementary mathematics. It is designed to have the student commence work on the foundation concepts, rules and methods of basic mathematics. The main aim of this course is to provide a refresher in the fundamentals of basic mathematics, which are necessary to develop mathematics as a unified body of knowledge. Topics covered in the course include number types, operations with numbers (including rules of precedence), percentages, introductory algebraic manipulation, introductory statistics, exponents (indices), solving algebraic equations, coordinate geometry of a straight line and units and their conversions. These are the concepts that form the base for the hierarchical development of mathematics.

Transition Mathematics + (TM+)

TM+ is an intermediate mathematics course. Students are able to tailor the course to their future study plans by completing core and optional modules. TM+ is designed to follow on from a study of introductory mathematical concepts, such as TM1. The course has been developed to give students introductory knowledge of various mathematical topics including statistics, standard deviation, probability, absolute values, inequalities, simultaneous equations, quadratic equations, functions, logarithms, trigonometry, geometry, variation, ratio, proportion, financial mathematics, annuities and series and sequences.

Transition Mathematics 2 (TM2)

TM2 is a technical mathematics course, designed to follow on from TM+. It is the required preparatory course for the study of first year tertiary mathematics in applied science and engineering. TM2 includes additional algebra, matrix algebra, trigonometric ratios, trigonometric functions and their graphs, vectors and oblique triangles, plane analytic geometry, derivatives, applications of the derivative, integration and applications of the integral.

Depending on the current level of mathematical competency (assessed through an entrance test) and undergraduate requirements, students can enrol in either TM1 or TM+. Students requiring the TM2 prerequisite must complete TM+ first.

Approximately 1500 students were enrolled in a bridging mathematics course in 2010, with 66% being enrolled in TM1. As a result of the demand in students wishing to gain entry into engineering and science degrees, the enrolments in courses have been continuing to increase. Accommodating the diverse mathematical backgrounds of such a large cohort of students, especially when the majority have either not studied for a considerable length of time and/or have completed very little mathematics at a secondary level, is extremely difficult. This is a major issue faced by the MLC and is exacerbated when courses are delivered externally, which is the only mode of delivery for TM+ and TM2 [5]

One way to support student learning is through the provision of comprehensive learning packages for the bridging mathematics courses. These consist of a Course Outline, a detailed Study Guide (consisting of a course overview, sample tests and module tests), an MLC textbook (consisting of detailed explanations, worked examples, exercises, review exercises and full-worked solutions). These extensive resources, coupled with on-campus lectures, have been found to provide the necessary support for

internal students. Unfortunately, although external students are able to contact a lecturer via phone or email, they often find it extremely difficult to learn from text-based materials, especially as “the nature of mathematical sciences dictates that students need to hear the instructor explain the concepts and ideas” [6]. In order to actively engage external students and provide the support necessary for success, the MLC has utilised the benefits afforded by technology.

Technology

The use of technology is becoming a significant element in the teaching/learning context. However there is an increasing onus upon the role of the lecturer to ensure that students are able to interface with technology in a meaningful and purposeful way. The effective integration of technology into mathematics classes can have many positive effects, including improved attitude and increased engagement with mathematics, but these positive effects are dependent on how well the technology is used [7]. According to Pomerantz and Waits [8] “The medium is the message”. Individual impressions are shaped by the nature and form of the technologies used to encode and decode experiences. The technology conveys the critical and dominant information through its role in influencing the views, ideas and attitudes of the user. This principle is significant in the development of programmes which need to consider both the affective and cognitive domain of students’ learning. A multi-sensory approach provides a variety of mediums appropriate to a range of learning styles and allows students the opportunity to interact with the technology at their own level. A range of media, utilising the present defining technologies, fosters and enhances ways of thinking and the learning generated.

The MLC has embraced technology as a powerful learning and teaching strategy. Utilising Tablet PCs to create resources for Electronic Study Guides and electronic marking has facilitated the active learning, engagement and subsequent success of bridging mathematics students [5].

What is a Tablet PC?

The Tablet PC is essentially a laptop computer that enables the user, through pen technology, to annotate (write) on the screen. There was much excitement surrounding the technology in the late 1980s, reaching a peak by 1991 [9]. It was envisaged that this technology would eventually replace the mouse and keyboard, but they were difficult to use and the handwriting recognition was inadequate [9]. Fortunately, improved computer technology has resulted in greater functionality with the newer versions. Tablet PCs are now lighter, more efficient and more affordable; processors are faster; resolution is finer; and the handwriting recognition software has been vastly improved [10]. Additionally, no attempt has been made to replace the mouse and keyboard in the current Tablet PCs and although handwriting recognition is still a feature, “digital inking” (allowing the user to annotate on the computer using a stylus or pen) proves to be its strength.

Electronic Study Guide (ESG)

Hargreaves and Jarvis [11] explain learning as “a planned process to modify attitude, knowledge or skill behaviour through learning experience to achieve performance in an activity or range of activities”. The ESG contains the entire set of course resources and can be accessed by the learning management system and/or a CD/DVD. The ESG enables students to easily navigate their way through the course, all

with the click of a mouse. It is designed to provide students with weekly directions to keep them on track with their studies. As the student works through the week's instructions they are able to read the relevant sections of the textbook; watch instructional videos; complete and check textbook examples; and when they finish a module they can easily access the corresponding sample test (with worked solutions) before accessing and completing the module test required for submission. This is facilitated by hyperlinks within the week's instructions (see Figure 1).

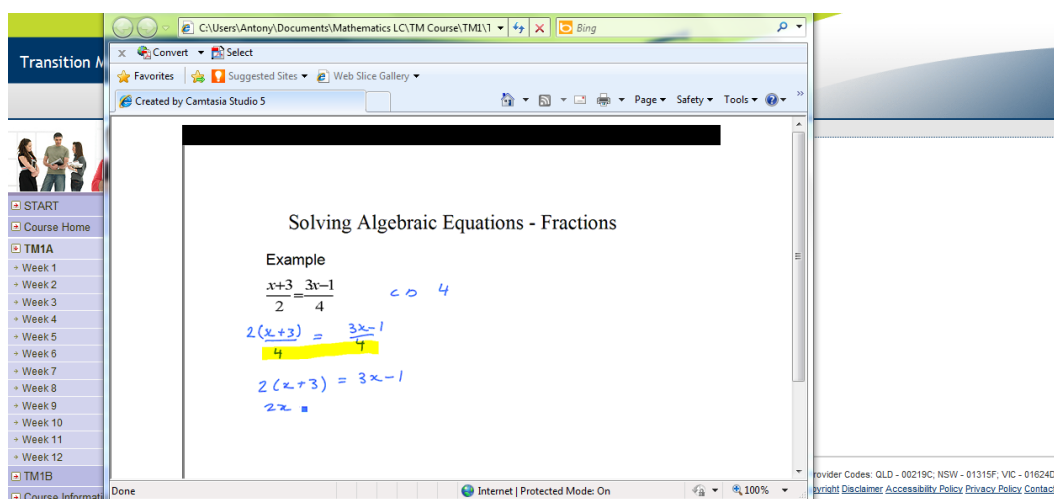


Figure 1. Electronic Study Guide.

Making external materials stimulating and meaningful is an on-going challenge and one in which the MLC has a particular interest. In the online environment, we have incorporated a number of innovations designed to provide the same kind of scaffolding that is available to students in the face-to-face situation, including PowerPoints® or slides, discussion boards, readings and instructional videos. Students greatly appreciate the ease afforded by the ESG, as witnessed for example in one student's response, who notes:

I'm really enjoying maths - the program explains concepts really well (TM1 external student, 2011).

Instructional Videos

“Rote computation and tedious algebraic manipulations have historically turned many students away from mathematics” [8]. Mathematics has been associated with the memorisation of number facts and formulas, algorithms and number substitutions in equations, drill and consolidation exercises and performing long, monotonous computations. Individuals whose motivation and application are frustrated by the tediousness and repetitiveness of this process are able to be actively engaged via instructional videos developed by the MLC. The instructional videos, developed using a Tablet PC in conjunction Camtasia®, that record the screen and sound, are embedded into the ESG, thus making it a very powerful tool. The videos enable the student to hear and see the mathematics unfold as if they were in a class situation. The lecturer not only explains the concepts and ideas but also the mental processes involved in problem solving. Robson, Abell & Boustead [12] highlight the importance of students being able to mentally plan a sequence of strategic decisions when forming a strategy for solving

equations.

Having the videos embedded into the ESG guides students through the course content in a similar manner to attending lectures or tutorials, with the added benefit of being able to revisit, replay and pause the instruction. Additionally, as well as being embedded into the ESG, there is a separate link to the videos which enables the student to view selected ones again without the need to remember the week of study in which they occurred.

The majority of students find the instructional videos to be extremely beneficial to their understanding of the material:

I found the videos an invaluable resource for me personally and I believe other external students would feel the same. The videos are the reason, I believe, why I have exceeded my personal expectations in the maths component of this course (TM1 external student, 2006).

The Tablet was great. It allowed us to see exactly how to solve the problem, step by step, and allowed the teacher to explain his thinking as he went along. It is always better to see it worked out in front of you than to look at the already made answer and try to decipher it. (TM1 student, 2010)

At all times the aim is to encourage students to take ownership of their own learning. Additional activities are provided for learners experiencing difficulty with concepts, as well as opportunities to access specific resources and individual tutoring. Short personalised videos can also be created by lecturers to further assist external students experiencing difficulties. These videos are quick and easy to develop and can be created in any location, even away from the office. When students are having difficulty with a particular concept or problem in the textbook, they appreciate being able to email their lecturer for help and receive a short video explanation in return:

That is excellent. Thank you so much. (External undergraduate student, 2011).

Thank you, it all makes so much sense when you do it! I certainly didn't expect an answer on a Saturday, boy am I glad you were at work...That did help a lot, thanks again, (External undergraduate student, 2010).

Evaluating the effectiveness of the instructional videos

Skills for Tertiary Education Preparatory Studies (STEPS) is a CQUniACCESS program that commenced in 1986. In 2008, students enrolled in the external STEPS programme were provided with video instructions for the completion of all sample assessment tests via the ESG. Students were not given instructions pertaining to the navigation of the ESG, nor were they specifically directed to the sample assessment test videos. At the end of the course the students were sent a questionnaire. The researchers were not only interested in the students' attitudes toward the videos, but also if they were easy to locate and view, thus enabling the MLC to evaluate the effectiveness of the ESG in assisting students to prepare for assessment.

With a 34% response rate, the results from this study confirmed the MLCs suspicions that students not only found the videos benefited their TM1 studies but, due to the format of the ESG, they were easy to find. The majority of students thought the videos were easy to locate but, although they also regarded the content as clear and easy to understand, they tended to feel more explanation would be beneficial. When asked what aspects of the recordings they found most helpful, 97% of the students said seeing the solutions worked step-by-step and seeing the correct setting out. 77% of the students found being able to hear as well as see the solutions was helpful. Surprisingly only 53% of these students thought the videos added a personal touch, thus adding to their

enjoyment of the course.

Students did comment on the quality of the videos. They found the background noise that was present on some of the videos was distracting. Also the quality of the instructor's voice played a part in their satisfaction. This demonstrates the importance of quiet recording spaces and good quality equipment. For example, microphones need to be strategically placed so that excessive breathing is not recorded. Approximately 50% of students liked the ability to replay sections of the recording until they understood the concept.

The recordings are extremely useful, they deliver explanations and you can replay back until you *understand* (student comment).

In 2010/11, a full suite of instructional videos were created for TM+, and TM1 videos have been enhanced where required. Due to the advancements in recording technology and the subsequent quality of the instructional videos, the MLC are planning to evaluate the effectiveness of the resources again. Judging by the following comments from TM+ students, it is anticipated that the satisfaction level of students will have increased as a result:

I loved having the videos for the first few modules of TM+ and really miss them now I have moved on. The videos really helped me understand some of the more difficult concepts and it was great hearing the explanations and solutions, rather than just reading them from the book (TM+ external student, 2010).

Your course definitely had the support that is required and your videos were very helpful. Hopefully that kind of support will be available for TM2 soon (TM+ external student, 2011).

Formative Assessment

Davis and McGowen [13] found formative assessment was the key to student success in mathematics. At the end of each module of study, the ESG directs students to complete a formative assessment test (End of Module test (EOMT)) covering the content of the module just studied. Hattie [14] insists that in order to gain excellence in education the "teachers need to be aware of what each and every student is thinking and knowing". For external students, formative assessment provides the lecturer with an insight into the students' level of understanding so that they are better able to assist them. Students are invited to submit their tests via email, fax or mail. All tests are stored and marked electronically using a Tablet PC and returned by email. Using a Tablet PC for marking enables the lecturer to annotate the test and provide prompt quality personalised feedback.

Electronic submission, storage and the ability of markers to immediately access and mark tests through a shared drive, regardless of location, has reduced the turnaround time to process assessment from 1 to 2 weeks to 2 to 3 days. External students are the main beneficiaries of this efficiency, as they do not have to wait for their assessment to be submitted and/or returned to them via the traditional postal system. This process has an added advantage of allowing the lecturer to keep a permanent record of the student's exact submission as well as an exact copy of the feedback returned to the students. As students need to be involved in the learning process, they are given the opportunity to discuss their progress and formative assessments with the lecture and other students. It is recognised by Darkenwald and Merriam [15] that "Learning is more effective if the adult learner is actively rather than passively involved in the learning activity". Involving the student in their on-going formative assessment not only encourages engagement and collaboration between students and lecturers but also prepares students

for their summative assessment, which students greatly appreciate:

The quick turnaround of students' submitted work is also one of the key contributors to their success – being able to e-mail in their scanned work and then having it returned so promptly is a great initiative. Our WIST students have had a very successful year and we thank you for your contribution to their success, Regards Robyn Donovan (Coordinator, Women in Science and Technology, CQU, 2007).

Many thanks Sherie for the quick turnaround (TM+ external student, 2010).

Feedback

As the Tablet PC allows the lecturer to write on the student's test, personalised handwritten feedback (Figure 2) can be provided promptly, ensuring students receive feedback whilst the concepts covered are still fresh.

1. Solve each of the following for 'a'.

i) $-2a > -8$
 $a > 4$ ✓
 $a < 4$ ✓
 There are 2 solutions for absolute values
 \rightarrow look at pg 40 & 41 of textbook

ii) $4a - 1 > a + 8$
 $4a - a > 8 + 1$
 $3a > 9$
 $a > 3$ ✓

iii) $4 - 2a < 3a < a + 8$
 $4 < 3a + 2a$
 $3a < a + 8$
 $a = 7$ ✓
 only half of the solution

Figure 2. Extract of an annotated TM+ test.

The importance of feedback provided through formative assessment is not only an important part of the learning process but is also reciprocal. Through the submission of the assessment, the student provides feedback to the lecturer, who in turn provides feedback to the student through marking and annotation. If the student makes a mistake, the lecturer highlights the mistake and reworks the problem. This allows the student to see where they made their mistake and the correct working. Providing handwritten feedback is not only more authentic, but also provides guidance as to the correct setting out for a solution [16].

Submission of EOMTs therefore enables the lecturer to monitor the student's progress and provide extra assistance if required. This feedback is not only beneficial to the individual student but the entire external cohort. Through the submission of EOMTs from multiple students, the lecturer gleans an indication of the general understanding of the mathematical content. This identifies shortcomings in the course content such that future modifications are able to be made to the course, including additional instructional videos, as required. Through careful scaffolding of ideas and on-going constructive feedback, many students over the years have been able to achieve levels of competency in mathematics that they would not have previously imagined possible.

Thanks for all your help and support Sherie, I couldn't have done it without you...and by the way you have made my day :) actually quite a few days...lol (TM1 student, 2009).

Conclusion

As universities aim to increase student enrolments, especially LSES and mature-aged students, efforts must be made to ensure students are adequately prepared and supported. It is not acceptable to increase enrolments if the result is to either have students fail or have the quality of degrees decrease in order to maintain retention. With the aid of technology, the MLC has been consistently creating and improving resources in order to enhance the success of all students.

The Tablet PC has enabled the MLC to provide extensive resources that actively engage external students and comprehensive and informative feedback in a timely manner. The ESG is both an innovative and adaptive form of technology which supports the teaching and learning process. The potential of this tool is evident and significant in terms of its application within the learning context. As well as positively influencing the students' attitudes towards mathematics and their work habits, it also affords them an understanding of the relevancy and application of mathematics. The ESG provides external students with a structured learning environment similar to that provided to internal students. Additionally, the inclusion of electronic and paper based study materials in learning packages provide students with multi-sensory mediums, thus catering for a variety of learning styles.

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Instructor Perceptions of Using a Mobile Phone-Based, Free Classroom Response System in First-Year Statistics Undergraduate Courses: Implications for Teaching Practice

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Student engagement at first-year level is critical for student achievement, retention and success. Research indicates that the use of Classroom Response Systems (CRS) is associated with positive educational outcomes by fostering student engagement and by allowing immediate feedback. Traditional CRS rely on special and often costly hardware (clickers). Often, special software and IT support is required as well, thus increasing the costs of implementation and use. In this study we explored, from an instructor perspective, the use of a low-cost CRS (VotApedia), which enabled first-year students to become anonymously engaged in a large-class environment by using their mobile phones to vote on multiple-choice questions posed by the instructor. The project was implemented at three Australian universities in first year undergraduate statistics classes. Using an action research methodology, instructors involved in the study collected qualitative data related to setup of the system, in-class delivery and instructor perceptions of student engagement. This paper presents the results of the study including the advantages/disadvantages of using VotApedia, practicalities for consideration by potential adopters and recommendations for the future from an instructor perspective.

Keywords: statistics, undergraduate, clickers, classroom response systems, VotApedia, CRS, teaching

Introduction

Student engagement, recognised as particularly challenging at first-year level, is critical for student achievement, retention and success [1]. Research indicates that the use of Classroom Response Systems (CRS) is associated with positive educational outcomes by fostering student engagement and by allowing immediate feedback [2]. CRS are defined by Bruff as ‘instructional technologies that allow instructors to rapidly collect and analyse student responses to questions posed during class’ [2, p. 1]. Traditional CRS rely on special hardware, often generically called *clickers*, to enable students to engage in voting. Typically, students are presented with a question and a list of multiple-choice answer options. Using their clicker device, students then select the best option from the provided list. When requested by the instructor, the results can be displayed to the class (and instructors) immediately. Barnett classifies the advantages of using a CRS into three groups: attitudinal, interactional, and pedagogical [3].

Student attitudes are improved because the use of a CRS is seen by students as fun and convenient [3]. In some cases, students’ attitudes were positively impacted because

the CRS was used for grading [3]. Improved attitudes have been shown to improve attentiveness of students [4] and improve class attendance [5], provided the CRS is used well [6].

Barnett sees the interactional advantages of a CRS in terms of student engagement and in giving students immediate feedback, a view that is supported by other scholars [3, 5]. Others have demonstrated that the use of a CRS increases student engagement [7, 8, 9], especially among those who are unfamiliar with large classes [10] as is often found among first-year students. Using a CRS to engage students has advantages over many other methods, such as raising hands, because the interaction is anonymous [11, 12, 13] and so students do not fear being wrong in front of their peers [14]. Importantly, this means that the use of a CRS particularly engages students who otherwise remain disengaged, such as students with 'lower class standing' [10] or students self-identified as reluctant participators [15].

Barnett sees the pedagogical advantages as occurring at a higher level (metacognition) as well as at more basic levels [3, 16]. Importantly, this empowers students to evaluate their own performance [15] and to monitor their own understanding of content throughout the course [17]. As a result, the use of a CRS has been shown to increase students' long-term retention of knowledge [4, 18, 19, 20] and to increase student achievement [17, 21, 22]. A further pedagogical advantage is that the use of a CRS provides immediate feedback to the instructor about specific topics where students lacked understanding [23], so that more or less instruction can be delivered as appropriate [24]. Some instructors were successful in incorporating CRS into assessment as well [25]. In addition, using a CRS is a useful method for implementing peer instruction [26], which has been shown to increase mastery of conceptual reasoning [27, 28], and agile teaching, where questions are used to teach and to inform the direction of the lecture rather than to test students [29].

All of these advantages are crucial to effective teaching and achieving successful learning outcomes in students, and increasing student engagement. However, practical difficulties may arise when using a traditional CRS in large classes: financial, technical and pedagogical [3].

Financially, the physical clicker devices are expensive to purchase. Some universities require students to purchase individual clickers, imposing a cost on the student (called 'expense shifting') and is often resented by students [12]. When clickers are used for assessment, the purchase of the clickers becomes a compulsory cost.

Technically, numerous studies report problems when using physical clicker devices [30, 31], which in turn lead to student frustration and the wasting of time in class. Further, the physical clicker devices may fail, and need repair, maintenance or replacement. For some CRS technologies (such as clickers using infra-red technology, which requires line-of-sight between the clickers and the receiver), issues of reliability may also be relevant if the room conditions are not optimal.

Pedagogically, students need to learn how to use the new clicker devices, and training is recommended [3]. The time taken to address this issue takes away from time spent on the course content, and may present a burden to teaching staff. In addition, students perceive this as a waste of class time.

More recently, CRS have been developed where students use their mobile phones rather than specialist hardware. Some of these systems are tied to specific publishers, such as Wiley's *ClickOn* system (<http://clickon.johnwiley.com.au/>, accessed 25 May 2011). Other mobile-phone based systems have a cost burden, such as PollEverywhere (<http://www.poll Everywhere.com/> accessed 25 May 2011; up to 30 responses are free). Another mobile-phone-based option is VotApedia (<http://urvoting.com>, accessed 29

May 2011), developed by Australia's Commonwealth Scientific and Industrial Research Organisation (CSIRO). VotApedia is a CRS not tied to any specific publisher, is free to use and implement, and so has the potential for widespread adoption. VotApedia is the system chosen for this study, because it is free to use and administer.

To use VotApedia, the instructor uses VotApedia's web interface to create a question with answer options. The instructor then displays this question to the class on a projected computer screen, together with the short list of answer options. Students select an answer, then vote by making a free phone call on their mobile phone to a given phone number, by sending a two-digit code to a given SMS number, or by voting via the web. The results are collated automatically on a server and can be displayed immediately (in real-time) on the screen. These results can then be used in the classroom to correct misunderstanding and generate further discussion, engaging students through an anonymous and safe process. In addition, students and instructor have access to immediate feedback on the progress of the lesson.

VotApedia has the potential to retain all the advantages of other CRS while overcoming many of the disadvantages, such as financial (it is free to implement and to use, so there is no financial disadvantage to the student or to the university), technical (since it uses the students' own mobile phones with which they are familiar, technical problems should be minimised) and pedagogical (the learning curve is minimal, so time will not be wasted on teaching students how to use the system). Nonetheless, difficulties do exist with using VotApedia.

On this basis, the overall aim of the present study was to evaluate whether VotApedia is a feasible and reliable CRS technology for increasing student engagement and student learning in first-year statistics classes at university. More specifically, this paper aims to identify instructors' perceptions of implementing and using VotApedia in first-year statistics classes; identify the facilitators and barriers to implementing and using VotApedia; and identify the practicalities that a statistics instructor may need to consider if deciding to use VotApedia.

Study Setting

The use of VotApedia in first-year statistics classes was investigated in this study at three Australian universities during Semester 1, 2011. Three of the authors were the instructors for these courses.

- SCI110 *Science Research Methods* at the University of the Sunshine Coast (USC). Enrolments were 731. USC is young regional university (established in 1996), with approximately 7300 students. Students in SCI110 were mainly from the health disciplines, plus some science students and a small number of engineering students.
- 6540 *Introduction to Statistics* at the University of Canberra (UC). Enrolments were 265. The University of Canberra is a former College of Advanced Education with around 10,000 students attending a single campus in the capital city of Australia. Most students in the course were enrolled in Sports Studies and Human Nutrition degrees. Approximately 4% were graduates of non-statistical disciplines studying in parallel with undergraduates.
- STA2300 *Data Analysis* at the University of Southern Queensland (USQ). Enrolments were 68 students on-campus. (Off-campus students did not participate in the current project.) USQ is regional university with over 24,500 students. The students in STA2300 were primarily from business, psychology and science disciplines with a small number from other disciplines.

A number of advantages exist in studying the use of VotApedia in this context.

Using VotApedia in large classes made it easier to identify any shortcomings of the technology in coping with the load of large numbers. Using classes of varying sizes and in different universities enabled instructors to identify more clearly the facilitators and barriers to implementing and using VotApedia. One of these classes is very large (USC), one is large (UC), and one is of moderate size (USQ). Further to this, each setting supported the evaluation of VotApedia within the context of statistics by using questions involving text, formulae and equations, and images, as these all appear naturally in statistics. From previous experiences of the three instructors, statistics is often met with trepidation by first-year students, so an anonymous method of engaging students was anticipated to be beneficial.

During the semester, each instructor kept notes of the strengths and weaknesses of using VotApedia, the barriers and facilitators to implementation, practicalities to be considered, and technical difficulties. A comprehensive online student survey also forms part of the evaluation of using VotApedia, but the survey was still open at the time of writing this paper, and will be reported elsewhere.

Data Collection

Data collection was based on an action research cycle (plan, take action, collect evidence, reflect) for each of the three instructors involved. Data sources included personal reflections, emails/conversations with colleagues and collaborators, class observations, and unsolicited feedback from students.

Reflections

Reflections on setup

As explained earlier, VotApedia has the potential to overcome many of the disadvantages identified with CRS, while maintaining the advantages. However, using VotApedia is not without its problems from an instructor's point-of-view, even though it was found that VotApedia was pedagogically very useful in the classroom. The major limitation is due to the interface and some idiosyncrasies in the way VotApedia is built.

For the instructor, the workflow needed to interact with the VotApedia web interface is not always clear, though the process becomes familiar and relatively easy after a couple of uses. Creating surveys in VotApedia can present practical problems. VotApedia offers six choices of survey types, each with pros and cons and different purposes. For example, a 'simple survey' is designed for single questions, while a 'questionnaire' is designed for displaying more than one question at a time. Initially, the interface for creating questions and answer options for a simple survey is easy to use. However, subsequent editing requires the use of a markup language (Figure 1). The markup language is the interface for the questionnaire and there are several peculiarities associated with its use that cause frustration, particularly in relation to the use of quotation marks and special characters. Even more frustrating, a question using quotations (for example) may appear to be formatted correctly in the preview, but then show errors when the survey goes live. Two of the authors discovered this in class.

One consequence of formatting restrictions is that questions may lack basic formatting and appear rather unprofessional. This all means that the creation of surveys to use in class must be completed with care. In particular, having a practice 'live run' in the instructor's office before entering the classroom is strongly advised.

Images (such as graphics, or screenshots of output from software packages) are easily included in questions, though the process is not obvious initially. Graphics may

be included with no difficulty. However, including mathematical equations can be problematic, even though a button is displayed that promises to offer such a service. After using this service, equations can appear nicely in the survey preview, but fail when the survey goes live. One workaround is to take advantage of the fact that the inclusion of images works well, and create equations as images (for example, using LaTeXEd <http://www.thrysoee.dk/laeqed/>, accessed 25 May 2011). While this is possible, the process of including an equation has become cumbersome.



Figure 1. Editing an example VotApedia question using the markup interface.

Regrettably, the display shown to the students contains a lot of unnecessary information that clutters the screen. Sometimes the instructor might wish to make the font sizes larger to ensure all students can read the question and phone numbers, but the font can only be increased so far because this extraneous information consumes so much screen area.

Reflections on engagement

From an instructor's point-of-view, the use of VotApedia has made the classroom a

more interactive learning environment. After posing questions, students have been encouraged to discuss the answers among themselves. While students are discussing, it is clear from a walk about the classroom that most students are actively engaged in discussing the question that has been posed.

In some lectures for SCI110 at USC, the instructor or a Research Assistant was able to count the number of students attending the lectures, while students were thinking about the VotApedia question. Participation rates could be computed, as the number of students who voted in each question was recorded automatically by VotApedia. The participation rates were available over six different teaching weeks (from teaching week 6 to teaching week 12) for the three lecture groups (A, B and C) of SCI110: data from three Lecture As, four Lecture Bs, and five Lecture Cs were available, sometimes with counts from more than one question per lecture. In total, data for 25 questions were available. The overall participation rate was 46%, with similar participation rates in each of the three groups (48% for Lecture A; 44% for Lecture B; 46% for Lecture C;). The instructor in SCI110 explicitly encouraged students to discuss their answers with those seated nearby before voting, so more students may actually have participated in the classroom discussion than is reflected by these percentages.

The opportunity to obtain instant feedback has also proved helpful. After answers were revealed, students often appeared eager to find the reason for the correct answer, especially when the correct answer was not obvious.

Reflections on use

In this section, the time spent on VotApedia in one of the study classes is summarised. The uses of VotApedia range from assessing learning and dispelling myths to demonstrating concepts, generating data for analysis and revising for exams.

In one of the classes (SCI110), one two-hour lecture each week was recorded for teaching weeks 1 through 11. Near the end of the semester, these recorded lectures were reviewed and the amount of time using VotApedia in each two-hour lecture was recorded. The lectures in teaching weeks 1, 2 and 10 used three VotApedia questions; the lectures in the other teaching weeks used two VotApedia questions. The time spent on each question obviously depended on the level of the question, and the amount of reading and thinking required. However, in general, the mean time spent per question was 7.1 minutes (median: 6.5 minutes; standard deviation: 2.2 minutes), including the setup, thinking and discussion time. The authors who used VotApedia believe that two or three questions in a two-hour lecture is a good balance between time spent and pedagogical gain, with two questions consuming about 15 minutes in a two-hour lecture.

The usual way to use a CRS is to assess learning by posing a question, and asking students to indicate their answers, perhaps after discussing among themselves. The results are generated, and the correct answer revealed (perhaps after a discussion of the options). As part of the authors' reflections on using VotApedia, different ways of using CRS were discussed.

VotApedia questions were used at the start of a lecture to review important concepts from the previous lecture, to ensure all students were ready for the new content. Sometimes, VotApedia questions were used partway through a lecture for the instructor to obtain feedback on how students understood the new material. Often, VotApedia questions were used during the mid-lecture break to promote discussion in the break, and to be able to refocus the class after the break by discussing the answers.

VotApedia can also be used to pose content questions, followed by a question about the students' level of confidence in their answer. A variation is to give answers with

levels of confidence built-in (for example, two answer options may be ‘The answer is 12, and I am very confident that I am correct’ and ‘The answer is 12, but I am not very confident that I am correct’). This gives the instructor more information about how the students understand the content.

Questions can be used to demonstrate the value of teamwork and talking with other students. A question is posed, and students are asked to vote. Students are then asked to discuss the answers with those near them. Students can then revote. In most cases, following the revote a higher percentage of respondents will select the correct answer, thus demonstrating to students the value of group work.

Questions can be used to demonstrate the content of the lecture, such as the effect of bias in question wording. In SCI110, three lectures were held every week to different groups of students. The instructor decided to ask students a different version of the same question in each lecture; *two* versions were intentionally constructed to be leading (in different directions). The following week the proportions voting for each answer were shown to all classes.

Questions can be asked in lectures to *generate* data for analysis. Responses to VotApedia questions can be used to discuss statistical concepts in class.

Implications for teaching practice

The acceptance of CRS has been relatively slow in Australian universities, despite their documented advantages and their large uptake in the USA [29]. Certainly, using VotApedia has distinct financial advantages over other CRS. There are no direct costs to the institution or to the students to use VotApedia.

The pedagogical disadvantages are minimised because students are not using specialised hardware, but their familiar mobile phones.

The technical problems with using mobile-phone-based CRS are quite different to the technical problems of using other CRS. Some students may not have a mobile phone, some may have poor reception in the lecture theatre, some may have forgotten their mobile phone, and some students may have no credit on their mobile phone (which prevents them from making a call, even if the call is free).

A series of issues and recommendations related to the use of CRS, most of which arise because of traditional clicker devices, have been identified in the literature [3, 30, 32, 33]. For example, the registration of the clicker devices should be streamlined [3], and student training to use the devices should be prepared and delivered at the start of each semester [3, 32]. Neither of these were an issue when using VotApedia.

The advantages of using a CRS, as given by other authors, are apparent when using VotApedia. From an instructor’s point-of-view, VotApedia can be made more instructor-friendly. However, many of these limitations are not apparent to the student. The nature and extent of VotApedia’s limitations mean that the authors recommend training for instructors before VotApedia is made widely available across universities.

Conclusion

In this paper we have collected the reflections of instructors on the use of VotApedia in first year statistics classes in three Australian universities. The instructors experienced a variety of benefits from using the CRS, whilst also acknowledging that they came across a variety of problems in using the software on a day-to-day basis.

In conclusion, VotApedia can be feasibly used in large first year statistics classes if the instructors are aware of, and able to work around, its limitations. On a personal level, the instructors involved noted changes in their skills, knowledge regarding the use

of CRS in general and VotApedia in particular. At an organisational level, instructors involved have a better understanding of the facilitators and barriers that could influence adoption of VotApedia. Availability of support and training for VotApedia implementation is recommended. When the online student survey is closed, the results will be analysed to assess student opinion of the technology.

One unrealised opportunity is for statistics instructors across Australia and New Zealand to compose and share VotApedia questions. Further research and larger scale implementations are recommended.

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Excursions to and from semantic oblivion

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Abstract

This paper is a follow up to an earlier article written by the author that introduced and illustrated notions of syntactic and semantic reasoning. Further evidence is provided, by means of recent pathological examples and experiences from the author's own teaching, that contrast different uses of syntax and semantics and highlight the fragility of language and mathematical formalism in communicating ideas and concepts. Syntactic reasoning is at the superficial end of the spectrum, "skating on the surface", and involves formal manipulation of symbols, simple rules and substitutions. Semantic reasoning is deeper, "diving down towards the seabed", and involves drawing conclusions from underlying meanings and heuristics. The author suggests that illuminating the tension between these two modes of reasoning may enhance approaches to successful mathematics teaching and learning. He continues his argument that increased consciousness of their interplay may lead to improved morale and attitudes, more robust learning outcomes and greater willingness to engage in challenging mathematical activities, particularly by students who may be regarded as weak or inexperienced. This paper also provides further historical context and outlines the origins of the terminology within the theory of formal languages. The examples also touch, in passing, on thorny and subtle issues of appropriateness of assessment, morale and the psychology of the herd mentality.

Keywords: mathematical reasoning, syntax, semantics

1. Introduction

It is common to experience frustration or feel demoralised when calculations or mathematical arguments inevitably go awry, make little or no sense, or appear to lack relevance or significance. It is unfortunate that what in fact may be natural states of incomprehension, or apparent "chaotic mindlessness", can become painful, have negative connotations and a tendency to undermine confidence and put students off mathematics, even permanently. People are not stupid simply because they cannot comprehend explanations, even when delivered with care and diligence by an experienced teacher. They are not hopeless mathematical thinkers just because they become lost or "frozen" in attempting to create their own mathematical solutions or arguments, even after feeling that they have already achieved a reasonable degree of comprehension. Mathematics is inherently difficult and the creative processes that lead to successful communication are fragile and easily corrupted. Language is the medium of communication, and, through syntax (which includes grammar and formal rules of manipulation and deduction), semantics (which includes meaning and any underlying ideas or heuristics) may be conveyed in various degrees of approximation. The nexus between syntax and semantics is poorly understood, and there has been a plethora of attempts to explore it (going back even to Frege (1892), and see, for example, Chomsky (1957, 1995), Heim and Kratzer (1998) and a discussion of the syntax-semantics interface in Escibano (1999)) and even develop a theory of formal languages that intertwine the two through the notion of a syntactic congruence (see, for example, Eilenberg (1974), a little more about which is explained below, for purposes of background for the terminology used in this paper). A recent book of Saul (2007) examines the role of cognitive psychology in attempting to explain the relationship between semantics, intuitions and failures or paradoxes

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associated with so-called anti-substitutions (in simple examples that ostensibly occur at the level of syntax).

Wigner (1960), in a physics context, writes about the “miracle of the appropriateness of the language of mathematics” in an influential paper whose title refers to the “unreasonable effectiveness of mathematics in the natural sciences”. But this apparent appropriateness or effectiveness comes at a considerable expense: it is easy to overlook hundreds or (in the case of calculus) thousands of years of mathematical evolution. What might seem natural or inevitable in hindsight is the result of many minds tilling the mathematical soil and making incremental contributions and adding flashes of inspiration to a creative effort that has been refined over many generations. Whilst we may look to certain landmarks or individuals as unlocking keys to the development of mathematics, one should not ignore historical context and framework in which discoveries are made. As Newton famously remarked: “If I have seen further it is because I have stood on the shoulders of giants.” The metaphor of standing on the shoulders of giants dates back to the twelfth century (see, for example, Merton (1965)), and is intended to pay tribute to an historical continuum. In the author’s opinion, the historical context of mathematics should have as much prominence in the classroom as the mathematics itself, at least in any introductory phase.

These remarks are also made because it is too easy (and especially dangerous in the context of teaching weak or inexperienced students) to take modern mathematical notation and terminology for granted and lose sight of the significance of a variety of conceptual advances that seem to us now quite trivial, but, in their day, were substantial breakthroughs. A symbol for zero was introduced in about 800 AD, and up until about the sixteenth century, solutions to quadratic equations took many lines and several cases to write down and explain (see Stillwell (1989) for an excellent historical account of this and the many variations, and Fitzgerald (2010), for a beautiful anecdote of the role of zero in putting him off mathematics at a young age).

Once modern mathematical language had established itself, combined with the axiomatic method initiated by Euclid (see, for example, Artmann (1999)), it seemed inevitable that leading mathematicians, such as Hilbert, would ask whether mathematics itself could be reduced to formal manipulation of expressions and axioms (see, for example, Ewald (1996)). Gödel (1931) essentially proved that interesting mathematics could not be trivialised in this way (his celebrated Incompleteness Theorem), and this is really the basis of all undecidability results (such as the undecidability of the word problem).

There is then an unbridgeable gulf between syntax and semantics. It should not be any surprise at all that students of mathematics, and practising mathematicians, want to minimise effort by using, wherever possible, syntactic or “formulaic” methods. These methods however are inherently fragile and inexperienced students frequently come unstuck using them. The established mathematician, by contrast, has a wealth of semantic knowledge and experience, combined with well lubricated technique, and is able to use syntax expertly to move quickly and economically through series of deductions and just use semantics at a few pivotal points in a calculation or extended argument. The author (see Easdown(2009)) contends that seeking to stimulate awareness of these two contrasting methods of reasoning should enhance mathematics learning. Even including the terminology “syntactic reasoning” and “semantic reasoning” in classroom practice and parlance may assist in highlighting levels of depth and degrees of importance of certain ideas or techniques.

At this point, we remark about the origin of this terminology from the theory of formal languages, used for example in theoretical computer science (see, for example, Eilenberg (1974)). Let Σ be an alphabet and denote by Σ^* the collection of all words over Σ , by which we mean formal strings of symbols from Σ . A formal language L is just a subset of Σ^* . Two words v and w from Σ^* are called syntactically congruent with respect to L , and we write

$v \sim_L w$, if substituting one for the other in any given context does not affect membership of L , that is,

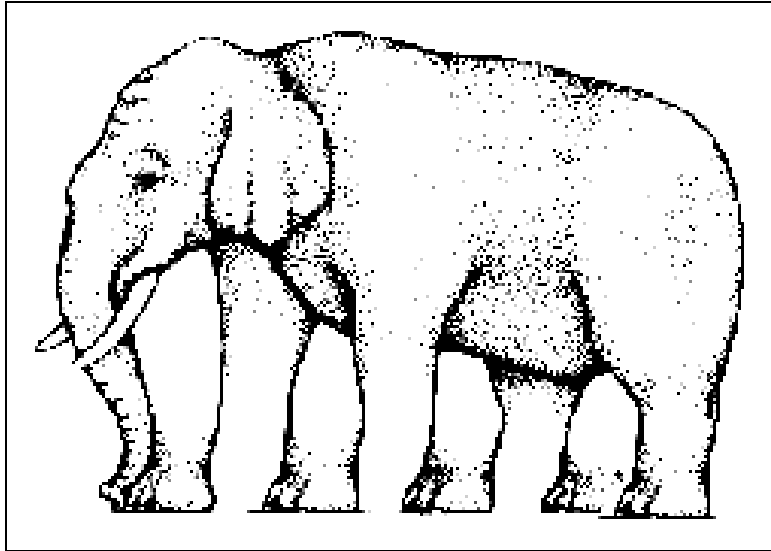
$$(\langle s, t \rangle \in \Sigma) \implies s \in L \iff t \in L.$$

This captures precisely the idea of u and v being equivalent “synonyms” with respect to L . For example, if L comprises all well-formed sentences in English, then all nouns become (syntactically) congruent, all verbs become congruent, but a verb and a noun will not be congruent. It is not difficult to modify L (for example, by only including fruit in the vocabulary of nouns used to make sentences) so that all names of fruit become congruent, all non-fruit are congruent, but a fruit and non-fruit are not congruent with respect to L , and then modify L again to distinguish, say, apples and oranges up to congruence. Modifying the language changes the syntactic congruence \sim_L on words from Σ . It then becomes fruitful to collectively study so-called streams or varieties of languages (see Eilenberg (1976)). The syntactic congruence classes with respect to a fixed L form a monoid under concatenation, denoted by $M_L = \Sigma^* / \sim_L$ and called the syntactic monoid of L . The relationships between formal languages and syntactic monoids are well studied and lead to an elegant and rich theory. For example, a language is regular (that is, built from singletons using boolean operations, concatenation and star) if and only if it is recognised by a finite state automaton, and this occurs if and only if its syntactic monoid is finite. Algebraic properties of the monoid M_L may be regarded as closely related to the underlying semantics of L , regardless of how L is described in terms of syntax or grammar. In this way syntax leads to semantics. We will not pursue this any further here, but the point is that the idea of simple substitution of words in context provides a test for syntactic congruence.

Whenever we perform mathematics by making a simple substitution, disregarding meaning, we are applying syntactic reasoning. Whenever we make a mathematical deduction using underlying heuristics or meaning, we are applying semantic reasoning. It becomes very interesting when errors creep in, that is, an incorrect substitution is made or an invalid heuristic applied. In a certain sense, all of the examples discussed in the remainder of this paper are pathological. But studying pathology is illuminating and strengthens our understanding of everyday phenomena, just as, for example, Oliver Sacks (1985, 1995) draws our attention to extreme examples of behaviour in neuropsychology, or a mathematician tests the boundaries of his or her theory using counterexamples (see, for example, Gelbaum and Olmsted (1990)). Errors and misconceptions are interesting and revealing, not just of a student’s current state of knowledge or understanding, but of the process of thinking itself, and strategies for tackling difficult or sophisticated mathematical problems. Rather than regarding the tension between syntactic and semantic reasoning as a nuisance or source of frustration, one can exploit the differences to create opportunities to enhance learning and expose weaknesses or gaps in understanding. Almost always, in the author’s experience, errors in reasoning tell us more than we imagined and their resolution make us more robust and creative in the long run.

2. Excursions to and from oblivion

The following figure of an elephant (downloaded from CoolOptical Illusions (2009)) with an indeterminate number of legs is a fine illustration of how slight perturbations of syntax (in this case the way feet are joined to legs in an outline of an elephant) can make an elementary question such as “How many legs does an elephant have?” difficult, if not impossibly difficult to answer, or even meaningless.



One could speculate how the artist came up with this figure. Possibly it was an intentional variation of the famous trident illusion, or it could be just that the artist misplaced the drawing of one of the hind feet, because of a suitable gap (all gaps are syntactically congruent!) and then proceeded to fill in some of the other gaps, and then realised the error leads to a pleasant illusion. This of course is an artificial example, but our students have no warning, in natural contexts, to help them recognise when something we tell them, or something they do themselves, is slightly “out of tune”, or when a seemingly innocuous question has not been properly formulated, and the associated anxiety and feelings of helplessness can be anything but pleasant, and further compounded by exam or (the equivalent of) stage fright.

In a recent course introducing calculus to students in agriculture, the author set an assignment, separated into parts, that explains the well-known *Rule of 70*. This rule tells an investor that approximately $70/i$ years is required to double an investment compounded at $i\%$ annually. For example, at 1% and 7% interest, one expects the investment to double in value after 70 and 10 years respectively. This is a good approximation for small interest rates. Students were asked to manipulate and apply calculus to the equation

$$2P = P(1 + i/100)^t$$

where P dollars is the amount of principal invested and t is the number of years for the investment to double. After one particular class, the author was visited by a contingent of students in distress because they could not get past an early step of the assignment, which was to eliminate P from both sides of this equation. The leader of the contingent was well spoken and articulate, and said “We have taken P away from both sides, to get

$$P = (1 + i/100)^t$$

and do not know what to do now.” The author replied: “Are you sure? What did you do to the left-hand side?” The student replied: “We took P away.” Author: “Hmmm...” The student went on: “If you take one P away from two P ’s you get one P . We don’t know what to do with it.” The students in the group, about six or seven of them, nodded in seemingly perplexed unison. Author: “But that is subtraction. What did you do on the right-hand side?” The penny dropped and they suddenly realised that they had confused division and subtraction together in the idea of “taking away”. The error was syntactic, in terms of formally manipulating symbols. However, there was probably a semantic component in being tempted to use subtraction: it is a common heuristic when introducing abstract variables such as P s and Q s, to think of them like apples and oranges. If you have two

oranges and take one orange away, you are left with one orange. This is such a powerful heuristic and appears to have infected the syntax of this simple first step (even though it is less effort just to cross the P out). This incident is also interesting, psychologically, because there were several students in the group asking for help, and it seems surprising that not one of them noticed the error, and all were eager for assistance. Magicians and other performers manipulate the fact that it is remarkably easy to fool or distract a large number of people. The group misconception here was unintentional, but the student spokesman probably had led the discussion prior to the group visiting the author, and he was confident and well-spoken. The correct cancellation of P leads to the equation

$$2 = (1 + i/100)^t$$

and then the next steps of the assignment required students to express in terms of i,

$$t = \frac{\ln 2}{\ln(1 + i/100)},$$

and finally use a tangent approximation (equivalent to the linear term of the Taylor series) to make the substitution

$$\ln(1 + i/100) \approx i/100,$$

and so deduce the *Rule of 70*:

$$t \approx \frac{\ln 2}{i/100} = \frac{100 \ln 2}{i} \approx 70/i.$$

The extended exercise involved formal manipulation of equations (or approximations) and substitutions, so was particularly ripe for syntactic errors. Others in the class found the fraction $i/100$ especially problematic and there were many stumbles and errors, particularly when differentiating and using the Chain Rule. The author believes that successful technique and understanding involving *fractions* lead to one of the key *threshold concepts* in mathematics, in the sense of Meyer and Land (2003, 2005). Many of our obstacles to teaching and learning introductory calculus would evaporate if students had already successfully passed through the “fraction portal”. Afterall, a derivative is nothing more than the limiting behaviour of a fraction or ratio (representing the slope of a secant of a curve). Unfortunately, talking about fractions has such a “primary school”, and therefore derogatory, connotation in a tertiary setting, that we don’t pay enough attention to it at university. (It seems ironic to the author that he also delivers fourth year Honours courses in which talented and gifted students at university learn about fields and modules of fractions, without any hint of stigma, involving precisely the primary school concepts that are required for a successful introduction to calculus.)

In the final exam for the same unit of study as the previous assignment example, students were asked to perform some very routine differentiations. Asked to find the derivative y' when $y = \frac{1}{3x}$, one student wrote:

$$y = \frac{1}{3x} = 3x^{-1}, \quad y' = -3x^{-2} = \frac{1}{-3x^2}.$$

This is an interesting and strangely beautiful answer and one can speculate about the student’s reasoning or thought processes. The author suspects (but is not absolutely sure) that, under exam pressure, this answer was produced hastily and with at best superficial and formulaic attention to detail. If one could have been a fly on the wall and asked the student at the time what he or she was thinking, most probably the phenomenon would

have been interrupted and the answer spoilt. The final expression for y' is correct, but the steps in the reasoning, if interpreted literally, become semantic nonsense. Most probably the student used syntactic substitutions in which the two errors cancelled out. If the student had bracketed correctly,

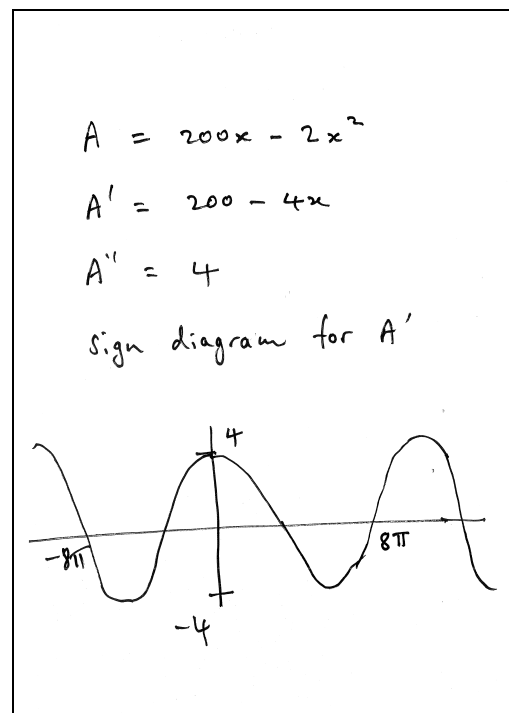
$$y = \frac{1}{3x} = (3x)^{-1},$$

then the most natural next step would involve the Chain Rule. It seems reasonable to assume the Chain Rule did not enter the student's mind at all and that only the formula for differentiating x^{-1} was consciously applied. If this is the case, then a very slight modification to the answer would deserve full marks:

$$y = \frac{1}{3x} = 3^{-1}x^{-1}, \quad y' = -3^{-1}x^{-2} = \frac{1}{-3x^2}.$$

This then would be quite a sophisticated answer, deliberately avoiding the Chain Rule, and providing evidence of full understanding of fractions, their manipulation and exponential notation. The author intends, in the future, to ask inexperienced students to analyse this example, and variations, as exercises. This example is interesting also in challenging the marker to appropriately and fairly assess the answer. Does it deserve 0, 0.5, 1 or perhaps even 1.5 marks out of 2? (It is not completely correct, regardless of how one interprets the student's reasoning.) Does one reward or penalise syntax or semantics? Under what circumstances does one have priority over the other? Should this be in the consciousness of the person designing the assessment?

Another question on this particular examination produced an extraordinary and unexpected answer, that surely would qualify for Dada art at its very best, though the author is sure that artistic expression was far from the student's consciousness at the time. The question asked students to use calculus to maximise a rectangular area A that turned out to have the formula $A = 2x(100 - x)$. One student wrote



The reader is left to ponder the conflagration of semantics and syntax involving diagrams, derivatives, sines, cosines, signs and cosigns. To the author's further astonishment, a second student in this particular examination attempted to use a sinusoidal curve to answer this question.

These examples so far involve students who are inexperienced or weak at mathematics. However, the author believes that the tension between syntax and semantics is a universal phenomenon concerning human communication and affects all of us, regardless of our experience or ability. The author set a difficult assignment question in abstract algebra for talented and gifted students in their fourth Honours year:

Assignment Exercise: Prove Blah Blah Blah.

What follows is an extended commentary surrounding Specimen 10 of Easdown (2009). The author received an elaborate answer from one particular student that, as it turned out (though not obvious from the way it was presented), separated into two halves connected by an “isthmus”:

$$(1 - e)m + em = 1 .$$

Each half was meticulously correct and involved sophisticated ideas and reasoning from the course, with good technique and correct semantics. The isthmus however involved a very simple “syntactic” cancellation error, typical of mistakes all of us make when routinely simplifying algebraic expressions. The brackets should have been expanded before cancelling, and the left-hand side becomes m , not 1, in which case the whole argument, as it turns out, unravels and falls apart. (The 1 on the right-hand side turned out to be crucial for the second half of the solution offered.) However there was a problem with the student’s semantics, because at the end of the second half, he claimed to have proved “Super Blah Blah Blah”, which in fact is false (only “Blah Blah Blah” is true). If he had thought a little about examples from the course, he would have realised that he must have fallen into error and then searched through and found his error and abandoned this particular solution. In fact, the density of the writing of the answer meant that the isthmus was buried somewhere in the middle of the argument and difficult to locate. The author awarded this answer 6/10, removing marks for the error and the failure to realise that that the final conclusion “Super Blah Blah Blah” was absurd. If this answer were an arithmetic calculation, for example, and it was not obvious from the context of the problem that the final answer was incorrect, then the error could be regarded as very slight indeed and the student might get 9.5/10. This was interesting also because the entire fourth year class was asked to peer review each other’s work. All of the student’s peers commented that this particular answer was worthless (0/10) on the basis that the conclusion “Super Blah Blah Blah” was ridiculous. Because of the density and length of the answer, none of the peers appeared to have the energy or inclination to try to locate the error or read and verify the careful reasoning in each of the halves surrounding the isthmus. This kind of example highlights the difficulties inherent in distinguishing between syntactic and semantic reasoning, the relative worth of industrious mathematical activity that leads to dead ends, and subtle issues about assessment and feedback.

We will finish with a story involving the author (and published in Easdown (1985)). Many years ago the author’s wife asked him to put the kettle on to make a cup of tea. He did so willingly, and a few minutes later the apartment rapidly filled with smoke followed by an explosion of flames. He had taken an electric kettle, filled it with water and placed it on an electric plate on the stove and turned the stove on (instead of plugging the kettle in to a power point and turning on the switch in the wall). His mind was distracted for some reason, and he managed to get inequivalent operations mixed up, made a simple syntactic error involving electrical equipment and nearly burnt the house down. His error was not dissimilar to the students confusing multiplication and subtraction in an earlier example,

though with potentially much more catastrophic consequences! One of the wonderful aspects of mathematics is that spectacular errors can be harmless and exquisite adventures that take place in the mind. We have much to learn from them and they are to be celebrated.

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The Use of the Interactive E-Book CAST in a First Year Business Statistics Paper

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The use of an interactive e-book instead of a traditional textbook in a large service Statistics paper is described. The rationale for employing an e-book, its implementation across different campuses and teaching modes, and the experiences of teachers and students are described. Future plans in light of these experiences are also outlined.

Keywords: e-learning, CAST, e-book, statistics training, teaching resources

Introduction

The authors of this paper are jointly responsible for delivery of a large 100 level service papers in Business Statistics at Massey University, New Zealand. Massey University has campuses in three New Zealand cities: Auckland, Palmerston North and Wellington, and also delivers papers in distance mode. The paper 115.101 Statistics for Business is a compulsory paper for all students in the Bachelor of Business Studies and Bachelor of Accountancy degrees and has been offered from all three campuses together with distance mode for several years. In 2011 there were almost 1500 students enrolled in the paper (see Table 1).

Table 1. Enrolments in 115.101 Statistics for Business by Campus and Teaching Mode, 2011

Auckland	Palmerston North	Wellington	Distance	TOTAL
467	256	108	666	1497

Up until the end of 2010 this paper used the Australasian version of a well-known textbook in business statistics, complete with the usual extras (website, question test bank, instructors' presentation slides etc). However several issues arose relating to the use of this textbook package (and others), namely:

- Electronic versions of supplementary material were shifted from CD to a secure website, which was not always stable and presented downloading problems for distance students with no broadband access;
- The Excel add-in which accompanied the textbook did not always produce correct output, and was dependent on the operating system used;
- Constant revisions of the text meant that core chapters would disappear from the printed version and be available only electronically, requiring either the university or the students to print off material at short notice;
- Several solutions to key exercises in the text had major errors that caused confusion and consternation to students.

These and other issues suggested that either a better and more reliable textbook and

publisher needed to be found, or else an alternative approach altogether was warranted, one that would address the needs of both on-campus and distance students in an equitable fashion.

Electronic textbooks

Many electronic textbooks (e-books) have been published but most have a limited amount of dynamic and interactive material, their examples are not aimed at an audience of business students and they could not be customised to the requirements of our particular course. We therefore decided to create an e-book for the course based on the e-learning software framework CAST [1] since it already included a business statistics e-book with extensive interactive material and could be customised with exactly the topics that were required for the course syllabus. CAST makes extensive use of dynamic and interactive graphics, and most CAST pages include an animation, simulation or other sort of dynamic display to help explain key concepts.

The use of computer technology to produce dynamic graphics and interactive data displays has been recognised as a valuable tool to aid understanding and learning of statistical concepts. The American Statistical Association's 2005 "Guidelines for assessment and instruction in statistics education" (GAISE) report [2] included "*the use of technology for developing conceptual understanding and analyzing data*" as one of its six recommendations. Specifically the report stated that "*technology tools should also be used to help students visualize concepts and develop an understanding of abstract ideas by simulations*" (p.20) and that technology should be used "*...as a way to explore conceptual ideas and enhance student learning*" (p.21). The report specifically mentions:

- generate and modify appropriate statistical graphics;
- perform simulations to illustrate abstract concepts;
- explore "what happens if . . ." questions

as ways that teachers should use technology, and that

- interactive capabilities;
- dynamic linking between data, graphical, and numerical analyses;
- ease of use for particular audiences;
- availability to students, portability

are considerations to use when selecting technology tools. These suggestions have been followed up by others, e.g. Chance and Rossman [3] suggest that teachers should

- Choose technology to facilitate student interaction and accessibility, maintaining the focus on the statistical concept rather than on the technology;
- Use tools (such as Java) that allow quick, immediate, and visual feedback;
- Carefully design the learning activity to guide student interaction; and
- Build on the simulations throughout the course enabling students to see simulation as an analysis tool in its own right, while also reinforcing key ideas.

There has been little published evidence to support claims of improved student performance through using e-books instead of paper-based textbooks. Symanzik and Vukasinovic [4] taught one stream of an introductory statistical methods paper with a good paper textbook and another with an e-book (CyberStats) and found few differences in student performance or attitudes. However the above reasons for including dynamic and interactive material to explain concepts and our bad experience with a business statistics textbook led us to believe that our students would benefit from a customised e-book.

CAST

CAST stands for “Computer Assisted Statistics Textbooks” and is a collection of electronic textbooks (e-books) that are designed to be read using a web browser rather than being printed.

Figure 1 shows a typical CAST page. After some text about a quadratic model for trend, an example is described and an applet allows students to investigate the scope of quadratic models by dragging the three red arrows. A button displays the least squares fit.

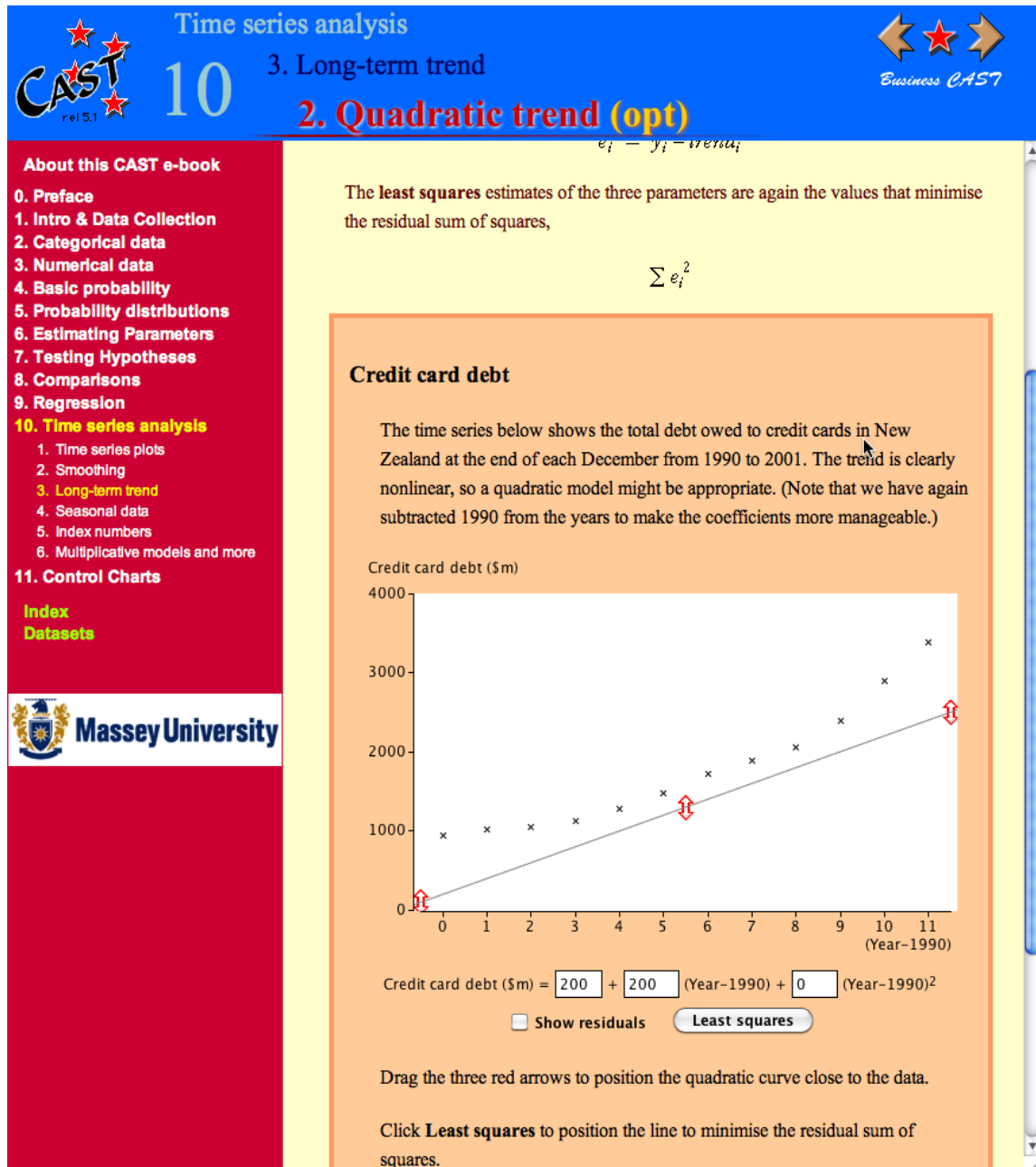


Figure 1. A typical CAST page about quadratic models for trend.

Figure 2 shows how CAST allows students to explore the differences between additive and multiplicative time series models. In the top applet, students discover that fitting an additive model to New Zealand visitor data does not capture the increasing seasonal variability. In the bottom applet, students fit an additive model to the logged

data and then a slider allows them to see how the fit is improved on the original scale. In either case a student can compare actual values and fitted values on the graph. Applets like these promote active learning in students and provide them with the opportunity to experiment with instantaneous feedback.

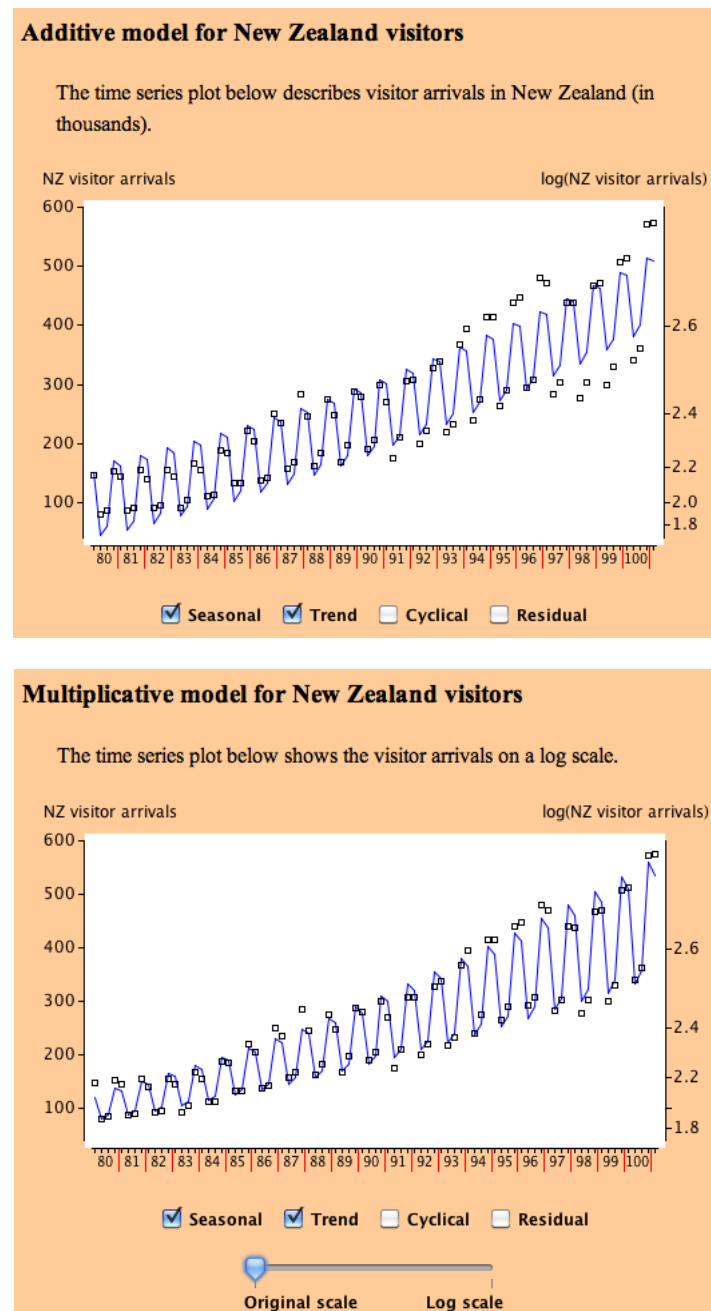


Figure 2. CAST applets showing additive and multiplicative models for time series data

CAST also provides a range of exercises [5], most of which use dynamic interactive displays, and which provide relevant diagnostic feedback. Figure 3 asks students to draw a Pareto diagram by dragging bars, then use it to find some proportion. Feedback is provided with the "Check" and "Tell me" buttons and many variations of the exercise can be generated by clicking "Another question".

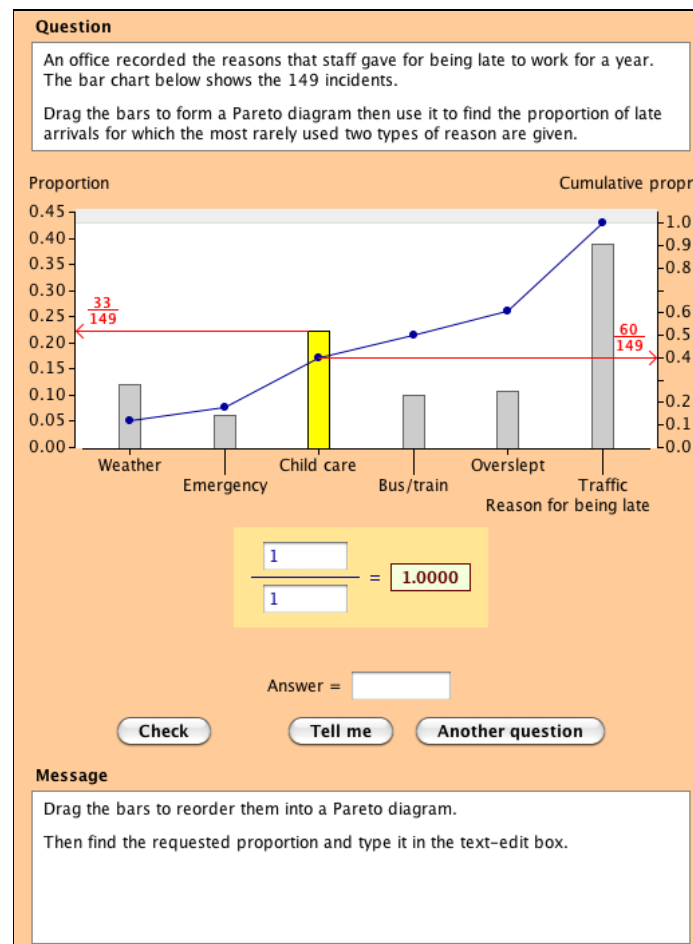


Figure 3. An exercise in CAST requiring dragging and a simple calculation.

CAST may be used over the web by users with fast internet, but it may also be downloaded, or installed from a CD/DVD to run locally without need for an internet connection. It can be used without charge under a “Creative Commons Licence”. The latest version (currently 5.1) can be downloaded from <http://cast.massey.ac.nz>.

The public release of CAST contains three introductory statistics e-books, including one with business scenarios and data, plus other advanced e-books. Its design makes it relatively easy to create a customised e-book using material from the core e-books and any required new topics.

Course Design and Materials Dissemination

The content and learning outcomes of the paper Statistics for Business had been determined in consultation with the College of Business at Massey University, so the course team created a customised electronic textbook based on the core business CAST e-book which conformed to these requirements. Some advanced material was deleted, other pages were rewritten and sections about probability, index numbers and multiple regression were added. Where possible, examples and dynamic diagrams used datasets that were relevant to business students, in particular datasets using New Zealand or Australian data.

For data analysis, custom-designed Excel templates were created to replace the Excel add-in supplied with the previous textbook. These templates did not require the Data Analysis toolpak and so could be used on Macintosh computers running Excel as well as on Windows.

Study and Administration Guides were provided on the course web site. PDF

versions of all e-book chapters were also generated. Although these could only contain static versions of the Java applets, they were made available to students as an alternative format that could be printed out and referred to in situations where computer access might not be available.

Distance students were sent copies of the CAST installer and e-book together with other materials on a CD, in addition to printed copies of the Study and Administration guides. All course materials were also accessible on the course web site and internal students were directed to it.

Apart from an initial meeting at the commencement of the project, the team worked by email and telephone across the three Massey campuses to produce the final version of the CAST e-book and the other course materials.

Student Feedback

The CAST e-book was trialled over the Summer Semester 2010 in two offerings, an internal offering at the Auckland campus of Massey University and in distance learning mode. Informal communications indicated that there were very few technical problems with installation and use of CAST. Each offering was surveyed using a standard university-wide online questionnaire that could not be customised with specific questions related to CAST and response rates were low. For example, only 17 out of 75 Auckland students responded to the survey.

Comments about CAST in the questionnaire fell into two groups, with some students loving it and others hating it. Positive comments included *"The interactive online book was great"*, *"Having the online interactiveness, with different questions within the same page. You could see the differences and try and see where you are going wrong"*, *"Thank God, you got rid of erroneous textbook. The CAST CD is great way to learn"*. Negative comments included *"For me personally, I like the methodology of following the concepts of a written textbook, where each chapter builds on from the next progressively"*, *"I still prefer to read a book as it can be taken with me anywhere. When you are studying while working full time you need to be able to take any opportunity to fit small amounts of study in"*, *"I found CAST extremely hard to use. Not the navigation part but just having to use computer screen all the time. I ended up printing all sections but had to keep coming back to PC to do exercises which was a nuisance"*.

Instructors at the Auckland campus noticed that students with problems in using CAST tended to be mature students, which suggests that familiarisation with electronic learning technologies may be a factor.

A model of learning such as that of Felder and others might explain these differences: those that like CAST may be more active and visual learners whereas those who dislike it may be more reflective and verbal learners – see e.g. Felder and Silverman [6].

Student feedback in Semester 1 at the start of 2011 was similar to that received over summer.

Discussion

Introduction of the CAST e-book has addressed many of the issues identified with the previous printed textbook packages, including the ability to fix errors or deficiencies quickly and efficiently.

Nelson [7] states that "One of the most challenging barriers to e-books is cultural resistance. For those who grew up with paper books (p-books) and always read from p-

books, switching to e-books is a bit uncomfortable for anything more than reference purposes." but then notes that this cultural resistance "will likely wane". Cutshall, Mollick and Bland [8] found in a survey of business students who used an e-book that 28% preferred web-based homework and e-textbook over paper materials, 50% preferred the paper materials and the remaining 22% were neutral. The mixed reception of our students supports these findings.

The lack of printed material that could be studied away from their computer was certainly an issue for some students. In Semester 1 of 2011, students were advised of alternative printed textbooks that could be used as supplementary study material and it was noted that small but significant numbers of internal students brought texts to lectures and workshops.

The CAST e-book was also provided as PDF files for printing, but this was a poor substitute, with too much paper used for static pictures of diagrams that are only effective in dynamic form. To be effective on paper as support for the e-book, the pages needed to be rewritten in summarised form. We are currently producing summarised versions of all e-book pages both as an extra for on-line study and to be the basis of a PDF version of the e-book for printing.

Conclusions

Survey results to date suggest that there may be two groups of learners: those who like CAST and those who dislike it. This may be due to incomplete survey data or may be an effect that could be explained using models of learning. Nevertheless we believe that e-books like CAST provide a customable and responsive active learning environment for students and that continued development and feedback will provide an improved learning environment for students of all backgrounds and abilities.

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Student Difficulties in the Production of Combinatorial Proofs

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Abstract

Combinatorial proof, the art of counting a set in two distinct ways to prove a statement, is a technique which emphasizes conceptual understanding of a problem and encourages creative thinking. We identify four categories of student difficulties with this type of proof, and introduce the term *pseudo-semantic proof production* to describe the attempt to write a combinatorial proof by relying on the syntax of previously encountered proofs. We illustrate the categories of student difficulties and pseudo-semantic proof production with four case studies drawn from a preliminary study of combinatorial proofs written by students in an upper-division combinatorics course and a graduate-level discrete mathematics course.

Keywords: combinatorial proof; proof strategies; semantic proof production; syntactic proof production

1 Introduction

Proofs of combinatorial identities can be produced in at least two distinct ways. One is to verify the identity algebraically, using known algebraic formulas. Such an algebraic proof is generally straightforward and usually causes no issues for students with strong algebraic skills. The second way of proving, combinatorial proof, uses a “double-counting” strategy of counting a set in two different ways¹. In a typical combinatorial proof, one manner of counting a set is represented by the left-hand side of the identity, and another way of counting the same set is represented by the right hand side. Since these two ways of counting count the same set, they must be equal to each other, thus proving the identity. Often, a “How many?” question is posed, with two answers given: one shows how the left-hand side of the identity answers the question, the other shows how the right hand side answers the same question. This double-counting strategy (including the use of “How many?” questions) is used extensively by Benjamin and Quinn in their landmark book on combinatorial proofs [1]. In contrast to algebraic verification, combinatorial proof often requires understanding of the meanings of combinatorial identities, as well as creative and intuitive mathematical reasoning.

The underlying logic of a combinatorial proof is not particularly challenging - a set is counted in two different ways. However, to successfully write a combinatorial proof, one needs to be able to imagine two different scenarios for counting the same set. A standard example is the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. A standard combinatorial proof imagines how one might choose a subset of size k from a set of size n in two different ways. The prover might choose the subset without paying attention to any particular element in $\binom{n}{k}$ ways, as on the left hand side, or the prover might consider a particular element, and whether that element is included in the subset (which can be done in $\binom{n-1}{k-1}$ ways) or not ($\binom{n-1}{k}$ ways), as on the right hand side².

Weber and Alcock defined a *syntactic proof production* to be one based on the manipulation of known facts and definitions within the representation system of mathematical proof [2, 3]. In contrast, they defined

¹Other combinatorial strategies exist; for example, one can find a one-to-one correspondence between two sets. However, the double-counting strategy is very common for these proofs, and we will focus exclusively on this strategy.

²Combinatorial proofs are often written in less formal contexts, such as picking members of a committee or toppings for a pizza.

a semantic proof production to be based on instantiations of mathematical objects outside the proof representation system (e.g. examples, counterexamples, mental or physical diagrams) to which the statement applies, guiding the formal inferences that are drawn. Weber and Alcock [2] noted that both methods of proof production can be successfully used by students, but suggested that semantic proof productions were more likely to lead to correct proofs more efficiently. Weber, Alcock, and Inglis have, more recently, suggested that it is possible that either proof production strategy can lead to success, with most mathematicians tending toward one strategy more often than the other [4, 5, 3]. Further, Weber and Mejia-Ramos [6] have proposed examining each inference that occurs while proving a statement to classify each and observe the translations between representation systems.

Applying these definitions to proofs of combinatorial identities, we can identify algebraic proof strategies with syntactic proof production, as only the algebraic definition of a combinatorial formula needs to be applied and manipulated. We could attempt to similarly identify combinatorial proof strategies with semantic proof production, as successfully counting a set using two distinct methods requires understanding of the meanings of the combinatorial functions at hand and the creation of an appropriate example context, as noted above. However, the strategy of creating and using a context to give “identities” to the elements of an abstract set to prove such a statement is a standard combinatorial proof strategy. Certainly these contexts can range from being a simple set of numbers to something more everyday and familiar such as pizza toppings. Therefore, this strategy could be considered to be within the proof representation system for combinatorial identities. We could consider such a production of a combinatorial proof to be syntactic. Thus, the production of combinatorial proof does not exactly fit within the constructs of the syntactic - semantic framework as defined to date. While the production of the simple proof above requires understanding the meaning of the choose function $\binom{n}{k}$ as the number of ways to choose a subset of size k from a set of size n , such an understanding may or may not involve objects outside the proof representation system, e.g. a mental image of selecting members for a committee.

In our experience, the context of a combinatorial proof is generated from the meaning of the combinatorial terms and the prover’s experience, and is often based on implicit or explicit mental or physical models of combinatorial objects. For example, Benjamin and Quinn [1] make extensive use of diagrams, dominoes, and other objects outside the proof representation system in their combinatorial proofs. Therefore, we will take the point of view that combinatorial proofs tend to be produced semantically, rather than syntactically.

2 Methods

This study examined the written combinatorial proofs of students in two classes: a graduate level discrete mathematics course and an undergraduate/graduate level combinatorics class at a large public university. In both classes, the instructors tended to present semantic productions of combinatorial proofs, using instantiations of mathematical objects outside the proof representation system. In particular, both instructors used contexts such as the selection of members of committees, and explicitly used mental, verbal, and written models of combinatorial objects in developing proofs.

We examined the combinatorial proofs written by students in response to questions on a midterm exam and a final exam. Both classes asked students to prove the following identity on the midterm exam:

$$\sum_{k=0}^n \binom{n}{k} \binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

On the final exam, both classes asked students to prove the identity:

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3.$$

On both exams, students were required to write combinatorial proofs; algebraic proofs were not accepted. The students’ written work was analysed using discourse analysis [7, 8]. By examining the students’ written responses to these tasks, we identified students’ successes and difficulties with writing such proofs. We also attempt to infer how the student uses mental models and constructs outside the proof representation scheme – that is, how the student uses semantic proof production. While it is impossible to definitively state how a

student is thinking based solely on written work, we examined the written work for artifacts of the students' models and constructs.

In both classes, the instructors encouraged students to write their combinatorial proofs by asking a "How many?" question that could be answered by both the left and right hand sides of the identity³. This advice was not explicitly followed by all students; although, some students may have implicitly used this advice to think of a question that could be answered by both sides of the identity but did not include this question as part of their written proof.

3 Results

3.1 Quantitative Analysis

In order to analyse the overall success of the students in writing combinatorial proofs, we gave each proof a score from 1 (for very unsuccessful proofs) to 4 (for very successful proofs). We also recorded whether or not each proof explicitly included a question that could be answered by both sides of the identity. The results of this quantitative analysis are given in Table 3.1. These results suggest that students successfully wrote proofs both when they explicitly wrote a question and when they did not, but were more likely to be successful if they did write a question, especially on the midterm exam. However, these results are neither statistically significant nor robust; more study is needed in this area.

Table 1: Student proofs rated by level of success and written question.

Midterm	4	3	2	1
Question written	13	4	1	1
Question not written	11	9	11	12
Final	4	3	2	1
Question written	13	1	1	0
Question not written	16	12	9	5

3.2 Common Difficulties

As noted above, the instructors of both classes tended to present semantic proof productions; students in both classes were taught to use contexts such as committee selection and mental models of combinatorial objects. It is therefore unsurprising that all of the student responses attempted to use similar contexts and models. When students were unsuccessful in their proofs, the errors made tended to fall into four broad (and overlapping) categories. We identify these categories of difficulties here, and illustrate them in the case studies in the next section.

1. Language mimicking

Some students' proofs contained instances in which the language they used was copied, almost wholly, from a previously seen proof of a similar identity. While this does not necessarily, in and of itself, cause errors, some students used such "language mimicking" inappropriately, in which they appear to be attempting to copy the syntax of the argument without considering the argument itself or the context in which the argument is framed. Such language mimicking can accompany evidence of misunderstanding of combinatorial functions: it appears that when students have a misconception or flawed model of a combinatorial function, they may attempt to fill the gap in their understanding with previously encountered language.

2. Inflexibility of context

³Such a question is another object outside the proof representation system.

Along with language mimicking, we saw some evidence that some students may try to use the same context for all or most of their combinatorial proofs. This can lead to errors when that context does not lend itself well to the identity at hand. If the student is unable to modify or change the context, he or she can have difficulty writing proofs.

3. Misunderstanding of combinatorial functions

A misunderstanding of combinatorial functions is an obvious barrier to the production of combinatorial proofs. If the student has a misconception of the meaning of a combinatorial function, such as the choose function, it will be difficult or impossible for that student to create contexts that correctly interpret that function. Evidence of such misunderstandings include, for example, choosing elements for a “subset” that are not included in the larger set.

4. Failure to count the same set

Some students presented proofs that failed to meet the basic structure of a combinatorial proof, in that they did not count the same set in two ways. Most often, this took the form of introducing a construct to perform one of the counts, without taking that construct into account when performing the second count. For example, if a student used the context of selecting members of a committee and selected a chairman of the committee in one of the counts, but did not select a chairman in the second count, then these two counts do not count the same set. One counts the number of committees with a chairman, the other does not.

In other cases, students concluded the proof with a statement such as “Since the LHS=RHS, they count the same thing.” This statement highlights a logical flaw; the two sides of the identity are equal because we count the same set in two ways, not the other way around. Such an error could also be a case of language mimicking.

From our observations, students whose errors fell into especially the first two categories may have been attempting to produce a proof using the kinds of contexts and models used in semantic proof productions, without fully understanding them. To describe this type of proof production, we introduce the term pseudo-semantic. We define pseudo-semantic proof production as the attempt to engage in a semantic proof production process, but relying on the syntax of a previously encountered proof when faced with a term that the student cannot explain. This syntax can include the specific context of a previously encountered proof. Misunderstandings of combinatorial functions can be linked to language mimicking and inflexibility of context, and therefore to pseudo-semantic proof production. By this definition, a pseudo-semantic proof production represents a failed attempt at a semantic proof production, and does not constitute a successful method of proof production.

3.3 Case Studies

In order to illustrate these difficulties, we present case studies of the written proofs of our students: Daisy, Arlene, Russell, and Ishmael. These students demonstrated differing abilities and development throughout the semester. We highlight aspects of their work in relation to the two problems presented above. In these case studies, we have typeset the students’ proofs, but have attempted to preserve the original format of the proofs as much as possible, including students’ grammar and spelling.

3.3.1 Daisy

Daisy was a graduate student who was very successful in writing combinatorial proofs. Daisy did not exhibit evidence of any of the difficulties above; we present her work in order to illustrate a standard to which the other case studies will be compared.

Daisy’s written response for the midterm question:

How many ways can we choose a committee of any size from a group of n people where the committee has a subcommittee of m people?

LHS: First we’ll choose each committee of size k from n people. In $\binom{n}{k}$ ways. Then from these k people in the committee, we’ll choose the subcommittee of m people in $\binom{k}{m}$ ways. Since we’ll do

this for committees of all sizes, from 0 to n , we first apply the mult. principle for $\binom{n}{k}\binom{k}{m}$ ways to choose the committee of size k , then apply the addition principle to sum the disjoint possibilities for each different size. This is $\sum_{k=0}^n \binom{n}{k}\binom{k}{m}$.

RHS: First we'll choose the m subcommittee members from the entire group of n . This is done in $\binom{n}{m}$ ways. To fill the remaining spots for the committee of any size we survey the remaining $n-m$ people. Since each person will either be on or off the committee, there are 2^{n-m} possibilities. By the mult. princ. there are $\binom{n}{m}2^{n-m}$ ways to create the possible (work cut off).

The final sentence of Daisy's proof was cut off in our copy of her proof, but it is clear that Daisy wrote a very successful proof. She followed the advice of the instructors in writing an explicit question to help her organize her proof.

Daisy was similarly successful in her proofs on the final exam. Her response to the final exam question was:

Q: How many ways can we select 3 students from a student body of $3n$, where n are singers, n are dancers, and n are musicians?

LHS: From the total group of $3n$, choose 3 in $\binom{3n}{3}$ ways.

RHS: Case 1: Choose all 3 students from 1 group. We can choose the group in $\binom{3}{1}$ ways, then the students in $\binom{n}{3}$ ways. By the mult. principle, there are $3 \binom{n}{3}$ ways to do this.

Case 2: Choose 2 groups, then 1 student from the first group and 2 students from the second group. Since order matters, there are 6 ways to choose a ranked pair of groups and $\binom{n}{1}$ and $\binom{n}{2}$ ways to choose the students from those groups. Thus Case 2 is covered in $6(n)\binom{n}{2}$ ways.

Case 3: Choose 1 student from each group. by the mult. principle this is done in $\binom{n}{1}\binom{n}{1}\binom{n}{1} = n^3$ ways.

Since the cases are disjoint, by the addition principle there are $3 \binom{n}{3} + 6n\binom{n}{2} + n^3$ ways to do this. As the LHS and RHS count the same thing, they are equal.

Daisy broke up her group of $3n$ students into singers, dancers, and musicians. While not explicitly written, it is implied that these groups are mutually exclusive even if they would not necessarily be so in real life. Daisy's work provided a good example of highly successful proof production. She embraced the opportunity to "write your own story problem" as part of the proof production process. Further evidence of this was found in the widely varying contexts used by other highly successful students. In the remaining case studies, we will, in part, be comparing the proofs of her peers to this standard.

3.3.2 Arlene

Arlene was a graduate student who struggled with writing combinatorial proofs throughout the semester. Arlene did not write a "How many?" question to be answered on either her midterm or final exam. On the midterm, Arlene wrote the following proof:

LHS: On this side we are adding the number of ways we can select people to do k jobs. So of the n people we can select k of them that are eligible to do the job and then m of them that actually complete the job. Each person can do more than one job (that's why they're not taken out of the set).

RHS: On the right hand side we are choosing m people out of n to complete the job and 2^{n-m} is the number of ways we can have them do the job. (2^{n-m} is the number of subsets the empty set is the case where $k = 0$)

Arlene had previously seen at least one proof using the context of choosing people to complete jobs in class. However, Arlene did not seem to have clarified her context: in her first sentence, she seems to be implying that k people must be chosen to complete k jobs, but later states that a person can complete more than one job. Her parenthetical statement of "that's why they're not taken out of the set" seems to mimic

the language of certain problems of probability, in which objects can be chosen from a set without being removed. Another possibility (though still an error) is that she may be thinking of selecting k jobs from a set of n people, which would constitute a misunderstanding of the choose function. The idea of having “2 to a power” represent the number of possible subsets of any size is a common idea in combinatorial proofs. However, Arlene appears to be mimicking that language without attempting to connect it to her context of jobs. Finally, Arlene fails to count the same set on both sides, as her argument for the left hand side includes consideration of eligibility for the job, while the count of the right hand side does not.

For the final exam, Arlene showed no improvement in her understanding of how to write a combinatorial proof:

LHS: Of the $3n$ jobs we are choosing 3 people to do them.

RHS: This is the number of ways we could have 3 people do the first n jobs in $3\binom{n}{3}$ ways, we could have 2 people do the second n jobs in $6n\binom{n}{2}$ ways, and we can have one person do the 3rd set of n jobs in $n^2\binom{n}{1}$ or n^3 ways.

Arlene again chose the context of assigning people to do jobs to try to write her proof, possibly indicating an inflexibility of context. Her sentences were very short and gave little detail about how she is counting. For the left hand side of the identity, she seemed to interpret the choose notation as choosing “people” from a set of “jobs,” indicating a misunderstanding of the choose function. On the right hand side of the proof, she appears to be making three statements, one for each set of symbols that has been presented to her, but she gives little indication of how these ways of selecting people to do the jobs are counted in this way, or why the addition of the three sets of symbols is appropriate. She appears to be mimicking language encountered in previously seen proofs by choosing three people for the $\binom{n}{3}$ group, two people for the $\binom{n}{2}$ group, and a single person for the $\binom{n}{1}$ group of symbols.

Arlene’s two proofs represent, perhaps, our clearest examples of *pseudo-semantic* proof production. Arlene appears to be inflexible with her context, and uses language mimicking extensively.

3.3.3 Russell

Russell, an undergraduate student, was not successful in his proof on the midterm, but showed some improvement by the final exam. He gave the following proof on the midterm exam:

RHS Of the total of the n population we want to make a committee of m people. Then from the $n - m$ people a either in the committee or not in the committee.

By multiplication principle, this will tell us the total of m people how many are in m committee.

LHS From the n total population we want to choose a committee of k people. Then from that k group we want to choose who will be in the m committee.

As $k \geq 0$ to n . This will determine the total amount of people that will be in m for each different k .

By the addition principle, the summation of all the k will tells us of the n population how many k people are in the committee m .

Thus, since RHS is the same as LHS, they are the same.

Russell’s proof had several problems. Like Daisy, he used the common notion that a power of two can represent choosing people to be on or off a committee, but unlike Daisy, he did not have a clear idea of how this should be integrated into his argument. In fact, it was not obvious that he attempted to integrate it at all, since he already chose a committee of m people. Since this “on or off the committee” notion is a commonly used one, it is likely that Russell is mimicking the language of other proofs without understanding it fully. He also invoked the addition principle, for no apparent reason other than that a summation appears in the identity. Finally, Russell failed to count the same set; one argument counts committees of m people, the other counts committees of k people. Russell’s midterm proof appears to be another example of *pseudo-semantic* proof production.

By the end of the semester, Russell showed some improvement in writing combinatorial proofs. He gave the following response on the final exam:

Left Hand Side: $\binom{3n}{3}$

We have 3 groups of n people. We want to take 3 of them to form a committee. This tells us the way to get 3 people from a population of size of $3n$ people.

Right Hand Side: $3 \binom{n}{3} + 6n \binom{n}{2} + n^3$

$3 \binom{n}{3}$: Counts the number of ways we can get a committee of 3 people of n population. Multiplying by 3 provides us the ways we can get a committee of 3 from 3 different groups of size n population for $n \geq 3$.

$6n \binom{n}{2}$: Counts the number of ways we can get a committee of 2 from n population. We also

write this as $2n \binom{n}{2} + 2n \binom{n}{2} + 2n \binom{n}{2}$.

This subcommittee generates n_1, n_2, n_3 counting the different subcommittee making up from each n population 3 times when choosing a committee of 2 people from a population of n people. Multiplying by the $2n$ generates from the population of $2n$ people. So, $6n \binom{n}{2}$ is the possible combinations of 3 n groups to make a subcommittee of 2 people for $n \geq 2$.

n^3 : Counts the population of all total n groups and there permutations they can have.

Thus, by the addition property, $3 \binom{n}{3} + 6n \binom{n}{2} + n^3 = \binom{3n}{3}$.

Russell did not write an explicit question as part of his proof, but he did appear to be counting the number of possible committees of three people chosen from $3n$ people in two ways. Some of the phrases he used as part of his proof are difficult to parse, and in reading his proof, it is not always clear exactly what his argument means to say. Toward the beginning of the "right hand side" section of his proof, he wrote, "Multiplying by 3 provides us the ways we can get a committee of 3 from 3 different groups of size n population for $n \geq 3$." We believe that he was correctly invoking the multiplication principle, by first determining the number of ways to choose three people from a population of size n , and then determining the number of ways to choose one of the three groups of size n . However, this was not entirely clear from his written proof.

In the second section of the right hand side, Russell used a similar phrase: "Multiplying by the $2n$ generates from the population of $2n$ people." His phrasing here was somewhat inscrutable; we believe that Russell may be trying to (correctly) communicate a choice of 2 people from a group of size n , and a choice of the remaining member of the committee from the remaining $2n$ people. However, this was far from clear, and it is certainly possible that Russell was, instead, simply trying to piece together likely sounding words to get his proof to "sound right," similar to Arlene. This interpretation is further borne out by his final case, in which he counts "permutations." We perceive this as evidence that Russell was mimicking the language of previously seen proofs in this part of his proof.

Although Russell was not completely successful in his proof on the final exam, his proof writing did show improvement. In particular, he showed improvement in connecting the two sides of the identity by counting the same set in two ways.

3.3.4 Ishmael

Ishmael was a graduate student who, like Russell, had difficulty writing combinatorial proofs on the midterm exam. However, by the final exam, Ishmael showed a marked improvement.

Ishmael's midterm:

Let $S = \{0, 1, 2, \dots, n\}$

L.H.S: Choose one person (k) of the group: $\binom{n}{k}$

Then, from the k person we choose sp. person: $\binom{k}{m}$

By the mult. princ, we have $\binom{n}{k} \binom{k}{m}$, since we have more than one set. Thus $\binom{n}{0} \binom{0}{m} + \binom{n}{1} \binom{1}{m} +$

$\binom{n}{2} \binom{2}{m} + \dots + \binom{n}{k} \binom{k}{m} = \sum_{k \geq 0} \binom{n}{k} \binom{k}{m}$

R.H.S: From the s group we choose a sp. person $\binom{n}{m}$, and from the committee we ask the sp. person if he in or off committee: 2^{m-n}
 By Mult. princ, $2^{m-n} \binom{n}{m}$

Ishmael's midterm proof showed a great amount of difficulty. He appeared to struggle with the meaning of the choose function. His use of the singular noun "person" throughout may be due to the fact that Ishmael is not a native English speaker, but his response indicates problems with understanding that the choose function indicates the number of ways to choose a selection. His phrase "since there is more than one set" may indicate a confusion with his context or his mental model of that context. Like Russell, Ishmael did not seem to have any clear idea why the addition principle should be invoked. Furthermore, Ishmael appears to fail to count the same set on the right hand side; his argument for the right hand side employs the same language mimicking ("on or off committee") as Russell, but otherwise is very difficult to understand. We believe Ishmael's midterm proof, like Russell and Arlene's, shows indications of pseudo-semantic proof production.

By the final exam, Ishmael showed improvement. His proof from the final exam is below:

Let S be a set with $3n \{1, 2, 3, \dots, n, \dots, 3n\}$ where n girls, n boys, and n teachers.

L.H.S: $\binom{3n}{3}$ Choose a subset of size 3 from the total set $(3n)$.

R.H.S: Case 1: $\binom{n}{1}$: Choose a subset of size 1 from n girls,

$\binom{n}{1}$: Choose a subset of size 1 from n boys and

$\binom{n}{1}$: Choose a subset of size 1 from n teachers.

By the Mult. princ. we get $\binom{n}{1} \binom{n}{1} \binom{n}{1} = n^3$

Case 2: $\binom{n}{3}$: Choose a subset of size 3 from n girls or

$\binom{n}{3}$: Choose a subset of size 3 from n boys or

$\binom{n}{3}$: Choose a subset of size 3 from n teachers.

By the addition princ. we get $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3}$

Case 3: $2n\binom{n}{2}$: Choose 2 subset of size 2 from n girls with a sp. girl or

$2n\binom{n}{2}$: Choose 2 subset of size 2 from n boys with a sp. boy or

$2n\binom{n}{2}$: Choose 2 subset of size 2 from n teacher with a sp. teacher.

By add. prin. we get $2n\binom{n}{2} + 2n\binom{n}{2} + 2n\binom{n}{2} = 6n\binom{n}{2}$

Consider all cases: case 1 + case 2 + case 3 $= 3\binom{n}{3} + 6n\binom{n}{2} + n^3$

Thus The RHS = The LHS

Ishmael's final exam proof was much improved; he clearly counted the same set in two ways, with the exception of "Case 3." In the third case, Ishmael was unable to see any reason for the multiple of n in the identity other than picking a "special person," which does not appear in the argument for the left hand side of the identity. This could be another instance of language mimicking, and of failing to count the same set. Nevertheless, this proof is a great improvement over his midterm proof.

We find it interesting to note that Ishmael's handwriting was very poor on the midterm exam, but his handwriting on the final exam was much clearer. Although we are not trained in handwriting analysis, it appears that Ishmael's poor handwriting on the midterm may indicate a lack of confidence, while the much stronger handwriting on the final exam shows increased confidence.

4 Discussion

By identifying some common errors made by students in writing combinatorial proofs, we hope to be able to guide students to be more successful in the semantic production of combinatorial proofs. In this study, we have identified four broad categories of difficulties that students may have when attempting to semantically produce combinatorial proofs: language mimicking, inflexibility of context, misunderstanding of combinatorial functions, and failure to count the same set. A common theme of student difficulties emerged from our analysis, in which students seemed to be doing more than mere pattern matching, yet failing to fully grasp the true meaning of what they are doing. We term this middle ground pseudo-semantic proof production. We

believe that students are making honest efforts to use the kinds of contexts and models that were presented by the instructors and are therefore attempting to engage in semantic proof production. However, some students appear to rely on language mimicking or on a single context when faced with difficulties. Others do not fully comprehend the meaning of $\binom{n}{k}$, and we believe these students attempt to make previously seen contexts fit their current identity without success.

We have already seen some success helping students avoid the fourth type of error by emphasizing the technique of writing a question to explicitly state what it is they are trying to count. Further work in this area, including obtaining student interview data, will help to develop other techniques to help students be more successful in writing combinatorial proofs.

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The Damped Pendulum

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Abstract

Viscous damping is commonly discussed in beginning differential equations and physics texts but dry friction or Coulomb friction and quadratic friction are not despite these frictions being encountered in many physical applications. One reason for avoiding these is that the equations of motion involve a jump discontinuity in the damping term. In this article we adopt an energy approach which permits a general discussion on how to investigate trajectories for second order differential equations representing mechanical vibration models having dry and quadratic friction. This approach is suitable for classroom discussion and computer laboratory investigation in beginning courses hence introduction of these frictions need not be delayed for more advanced courses in mechanics or modeling. Our method is applied to a pendulum model. One advantage of this energy method is that the values of the maximum deflections of a solution can be calculated without solving the differential equation either analytically or numerically, a technique that depends on only the initial conditions.

Key Words: dry friction, quadratic friction, dissipation, Coulomb damped pendulum, quadratic damped pendulum

1 Introduction

In beginning differential equations, linear models are usually fully developed using Newton's second law and Hooke's law. Forces resulting from the weight and stiffness are the first two terms usually accounted for. However, damping does not arise from a single physical phenomenon, it is a measure of energy dissipation in a vibrational model and plays an important role in the modeling of dynamical systems. There are as many types of damping as there are ways to convert mechanical energy into heat. The most often encountered frictions in physical models are that of dry or Coulomb friction when two surfaces rub together, and quadratic friction that results from air resistance or hydrological drag.

Studying damping arising from these frictions, using our elementary methods, can achieve two ends: firstly being the introduction of new, deeper and perhaps more relevant syllabus material and secondly, providing a demonstration of an integration between theory, numerics and graphical interpretation through the use of a computer algebra system.

Space does not permit the discussion of many examples, hence we focus on the damped pendulum model which takes the form

$$\ddot{x} + q(\dot{x}) + \sin(x) = 0 \tag{1}$$

where $q(\dot{x})$ is the appropriate dissipative term for the discontinuous Coulomb or quadratic damping. This model, being highly nonlinear, yields a rich test bed for our techniques.

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2 A Word On Friction

Usually, friction is a force that always opposes the direction of motion and thus is considered as a damping or dissipative force. In this article, we will only consider strictly dissipative oscillators. To interpret the frictional force for an equation of motion, with the correct sign to adjust for the direction of motion, we will employ the signum function $Sgn(\dot{x})$ defined by

$$Sgn(\dot{x}) = \begin{cases} 1 & \text{for } \dot{x} > 0 \\ 0 & \text{for } \dot{x} = 0 \\ -1 & \text{for } \dot{x} < 0 \end{cases} . \quad (2)$$

We could also agree to think of $Sgn(\dot{x})$ as given by

$$Sgn(\dot{x}) = \frac{\dot{x}}{|\dot{x}|}$$

whenever $\dot{x} \neq 0$ and is 0 when $\dot{x} = 0$.

In Coulomb or dry friction, the frictional force is proportional to the constant function 1. In this case the frictional force is

$$-cSgn(\dot{x}) \cdot 1,$$

where c (a positive constant) is the coefficient of friction. Note the minus sign in order to oppose the direction of motion.

Viscous or linear damping is proportional to the speed and is given by

$$-cSgn(\dot{x}) \cdot \dot{x} = -c\dot{x},$$

here c is simply called the damping coefficient.

For quadratic damping, the force is proportional to the square of the speed and has the form

$$-cSgn(\dot{x}) \cdot |\dot{x}|^2 = -cSgn(\dot{x})\dot{x}^2.$$

Thus the equations for a dissipative oscillator (with mass normalized to be 1) with Coulomb, viscous, and quadratic damping are respectively

$$\begin{aligned} \ddot{x} + p(\dot{x}) \cdot 1 + f(x) &= 0 \\ \ddot{x} + p(\dot{x}) \cdot \dot{x} + f(x) &= 0 \\ \ddot{x} + p(\dot{x}) \cdot \dot{x}^2 + f(x) &= 0 \end{aligned}$$

where $p(\dot{x}) = cSgn(\dot{x})$. To further simplify the notation we will write $p(\dot{x}) = \pm c$ where it will be understood that the plus sign is taken when the velocity is positive and the minus sign is taken when the velocity is negative. Notice in the viscous case, this differentiation is not necessary, since $p(\dot{x}) \cdot \dot{x} = c\dot{x}$ as usual.

3 Coulomb Damping

A common type of mechanical damping arises from dry friction. Charles-Augustus de Coulomb (1736 - 1806) won the 1781 prize from the Académie des Sciences for *Théorie de Machines Simples*, published in Paris in 1809 and therein pointed out the difference between static and dynamical friction. Due to this study, sliding friction is called Coulomb friction. Of course,

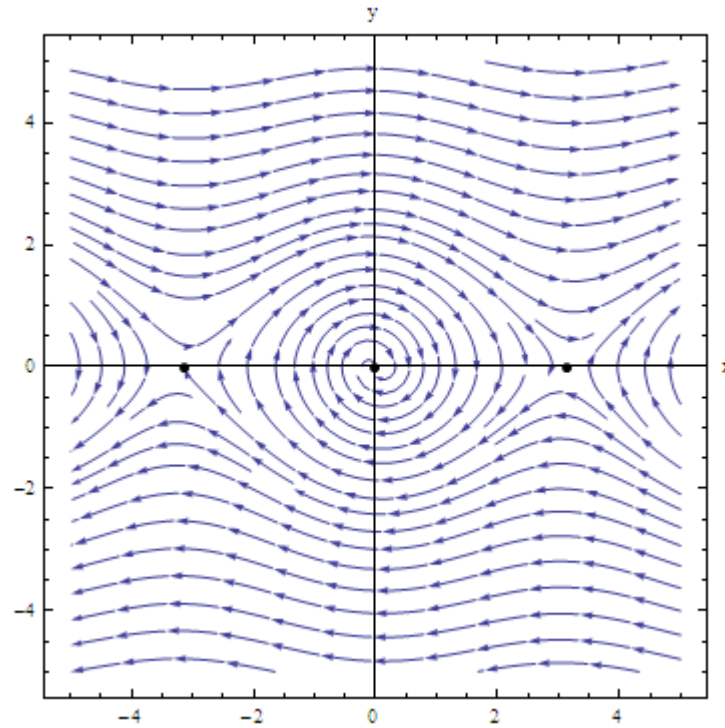


Figure 1: Vector field plot for the system (4), $c = 1/9$.

Coulomb is much better known for his foundational work in electricity and magnetism and in particular his inverse square law of electrostatic force.

The pendulum equation is found in most every beginning text. Because of its nonlinearity and exact solution involving an arcsine of an elliptic sine, the equation is usually given a linear approximation. For a derivation of the analytic solution see H. T. Davis [14] The undamped pendulum equation is investigated numerically together with various approximations in Fay [7]. The viscous damped pendulum equation is seldom discussed in any detail. For a detailed discussion of Coulomb damping see [8]

The Coulomb damped pendulum is a bit easier to investigate. This equation is

$$\ddot{x} \pm c + \sin(x) = 0. \quad (3)$$

Here we have normalized the mass of the system to be 1 and without loss of generality assume the coefficient of the sine term is also 1 (it is at most a time dilation to assume this). We can turn this equation into a 2×2 system by introducing the auxiliary variable $y = \dot{x}$, and obtain the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -c - \sin(x) \end{aligned} \quad (4)$$

We show a representative vector field plot for the system in Figure 1, where we have set $c = 1/9$ for visual convenience. It is straightforward to see the critical values of the system are $(\pm n\pi, 0)$ being attracting spiral points when n is even and saddle points when odd.

For a specific initial set of values $(x(0), y(0)) = (x_0, y_0)$, numerical methods can produce highly accurate solutions, but analytic solutions are much more difficult to obtain. However,

an energy approach yields quite a bit of information, and it is this approach (and a slight modification of it for quadratic damping) that we want to promote in this article.

Taking equation (3), multiplying both sides by $\dot{x}d\tau$ and integrating from $\tau = 0$ to $\tau = t$, we obtain

$$\int_0^t \ddot{x}\dot{x}d\tau + \int_0^t \pm c\dot{x}d\tau + \int_0^t \sin(x)\dot{x}d\tau = 0. \quad (5)$$

The first integral represents kinetic energy and the last integral represents potential energy since it depends solely upon displacement $x(t)$; the middle integral represents the loss of mechanical energy that is converted into heat and we call this term the *dissipation*. Integrating, we have

$$\frac{y^2}{2} - \frac{y_0^2}{2} + \pm c(x - x_0) + \int_{x_0}^x \sin(\xi)d\xi = 0. \quad (6)$$

Set

$$E(x, y) = \frac{y^2}{2} - \cos(x) + \pm cx \quad (7)$$

and call $E(x, y)$ the *energy function*. Rewriting equation (6), we have

$$E(x, y) = \frac{y_0^2}{2} - \cos(x_0) + \pm cx_0 \quad (8)$$

Note that the right-hand-side appears to be a constant so that kinetic plus potential plus dissipation is constant in the surface determined by the energy function and thus contours in the energy surface represent trajectories for the system with the initial values determining the contour constant. But the factor $\pm c$ means that as the trajectory crosses the x -axis ($y = 0$), the value of the contour constant changes.

We see that this energy function $E(x, y)$ is more than the usual "kinetic plus potential" since the dissipation term is present. From physical considerations, we know the sum of the kinetic and potential energies is not constant but decreases and these combined energies reduce as friction causes a "heat sink" and stable critical values are thus attracting spiral points for these oscillating systems.

3.1 Calculating Maximum Oscillations

We can calculate the maximum oscillations of any trajectory without having to solve the associated initial value problem, although having a picture of the trajectory is a help. Suppose we are given initial values (x_0, y_0) , and for the sake of argument assume $y_0 > 0$ (what to do when y_0 is negative will be obvious). The maximum amplitude of the first oscillation occurs when $y = 0$. This means we need to find the value x_1 where the trajectory first crosses the x -axis. This value is obtained by solving the equation

$$E(x_1, 0) = -\cos(x_1) + cx_1 = \frac{y_0^2}{2} - \cos(x_0) + cx_0 \quad (9)$$

for x_1 . In this case the trajectory begins at (x_0, y_0) and winds downward to the x -axis crossing at $(x_1, 0)$ and then, still winding clockwise below the x -axis, rises up to cross at $(x_2, 0)$. The value of x_2 is obtained by solving the equation

$$E(x_2, 0) = -\cos(x_2) - cx_2 = -\cos(x_1) - cx_1 \quad (10)$$

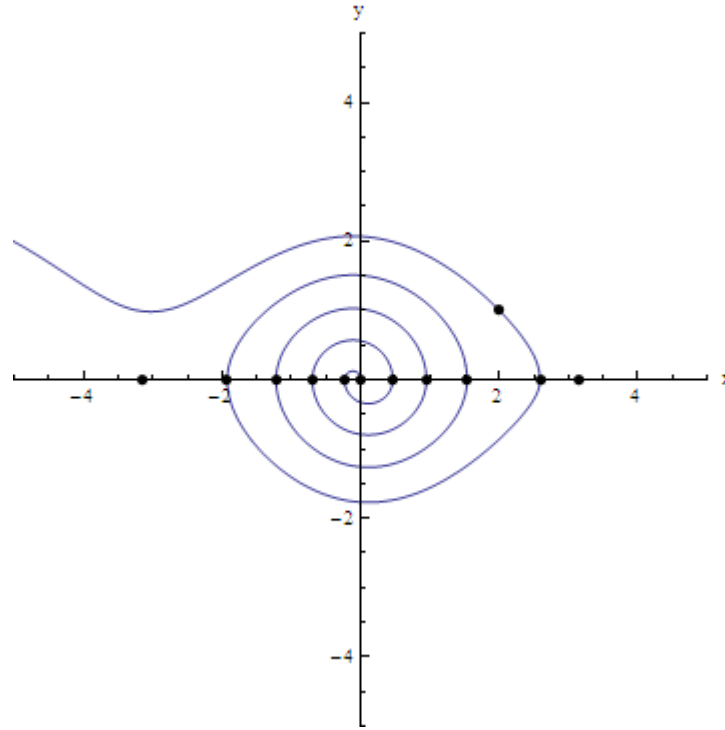


Figure 2: The full trajectory with initial point, critical values and crossing points.

for x_2 . Continuing winding, the trajectory crosses into the upper half plane and eventually crosses at $(x_3, 0)$ which can be found by solving the equation

$$E(x_3, 0) = -\cos(x_3) + cx_3 = -\cos(x_2) + cx_2 \quad (11)$$

for x_3 . In general, once the starting value x_1 is found, the iteration is: for i odd, solve

$$-\cos(x_{i+1}) - cx_{i+1} = -\cos(x_i) - cx_i \quad (12)$$

for x_{i+1} , and for i even, solve

$$-\cos(x_{i+1}) + cx_{i+1} = -\cos(x_i) + cx_i. \quad (13)$$

Solving these equations is quite easy using the Newton-Raphson algorithm which requires an initial guess, and choosing this guess is where having a picture of the trajectory is useful.

For example, if $(x_0, y_0) = (2, 1)$ and $c = 1/9$, then the crossing points are

$$\begin{array}{ll} x_1 = 2.58827 & x_2 = -1.92735 \\ x_3 = 1.53512 & x_4 = -1.22176 \\ x_5 = 0.94825 & x_6 = -0.69814 \\ x_7 = 0.46240 & x_8 = -0.235117 \end{array}$$

In Figure 2, the trajectory is shown with the initial point $(2, 1)$, critical values $(0, 0)$, $(\pm\pi, 0)$, and the crossing points $(x_i, 0)$ indicated.

4 Quadratic Damping

The assumptions for linearizing equations and incorporating viscous damping usually include small amplitude oscillations. Allowing larger deflections and perhaps the more realistic quadratic damping turns to be straight forward to discuss from a generalized energy point of view. A more detailed discussion on quadratic damping is given in [9].

The energy function for the quadratic damped pendulum equation

$$\ddot{x} + \pm c\dot{x}^2 + \sin(x) = 0 \quad (14)$$

is only slightly more involved to calculate than in the Coulomb case as we need an integrating factor to produce it. Multiplying equation (14) by the integrating factor

$$e^{\pm 2cx} \dot{x} d\tau \quad (15)$$

we have

$$\int_0^t e^{\pm 2cx} \ddot{x} \dot{x} d\tau + \int_0^t \pm c e^{\pm 2cx} \dot{x}^3 d\tau + \int_0^t e^{\pm 2cx} \sin(x) \dot{x} d\tau = 0. \quad (16)$$

We use integration by parts on the first integral letting $U = e^{\pm 2cx}$ and $dV = \ddot{x} \dot{x} d\tau$, to obtain $dU = \pm 2c e^{\pm 2cx} \dot{x} d\tau$ and $V = \dot{x}^2/2 - \dot{x}(0)^2/2$. so that, integrating from $\tau = 0$ to $\tau = t$, we have

$$e^{\pm 2cx} \frac{y^2}{2} + \int_{x_0}^x e^{\pm 2c\xi} \sin(\xi) d\xi = e^{\pm 2cx_0} \frac{y_0^2}{2} \quad (17)$$

where $y = \dot{x}$ and the initial conditions are (x_0, y_0) . The complicated integrals involving \dot{x}^3 , that occur from integration by parts and from the damping term, cancel out. The first term in equation (17) represents a generalized kinetic energy and the second term represents a generalized potential energy; generalized because of the factor $e^{\pm 2cx}$ and that equation (17) is two equations depending upon the sign of y .

In a bit more detail, we have for $y > 0$,

$$\begin{aligned} e^{+2cx} \frac{y^2}{2} + \frac{1}{1+4c^2} e^{+2cx} (2c \sin(x) - \cos(x)) = \\ e^{+2cx_0} \frac{y_0^2}{2} + \frac{1}{1+4c^2} e^{+2cx_0} (2c \sin(x_0) - \cos(x_0)), \end{aligned} \quad (18)$$

and for $y < 0$,

$$\begin{aligned} e^{-2cx} \frac{y^2}{2} - \frac{1}{1+4c^2} e^{-2cx} (2c \sin(x) + \cos(x)) = \\ e^{-2cx_0} \frac{y_0^2}{2} - \frac{1}{1+4c^2} e^{-2cx_0} (2c \sin(x_0) + \cos(x_0)). \end{aligned} \quad (19)$$

These equations determine the trajectory in the upper and lower half planes.

For a given value of c and initial conditions (x_0, y_0) , the amplitudes of the oscillations can be computed from equations (18) and (19) in exactly the same manner as above for the Coulomb damped case. To demonstrate this, consider the example with $c = 1/9$ and $(x_0, y_0) = (1, 1.5)$. We first determine that the first crossing point is $x_1 = 2.046591$ using equation (18) since $y_0 > 0$. Then interactively we solve for other crossing points x_i by solving

$$e^{-2cx_{i+1}} (2c \sin(x_{i+1}) + \cos(x_{i+1})) = e^{-2cx_i} (2c \sin(x_i) + \cos(x_i))$$

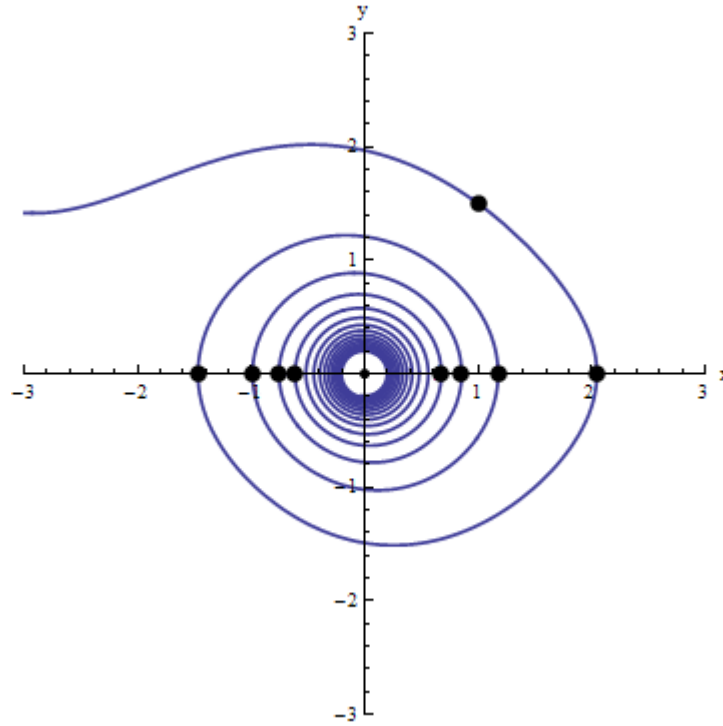


Figure 3: Trajectory, initial point (1 , 1.5), and first eight crossing points.

when i is odd, and

$$e^{+2cx_{i+1}} (2c \sin(x_{i+1}) - \cos(x_{i+1})) = e^{+2cx_i} (2c \sin(x_i) - \cos(x_i))$$

when i is even. The first eight crossing points ($x_i, 0$) are given by the values in the following table.

$x_1 = 2.046591$	$x_2 = -1.468825$
$x_3 = 1.176119$	$x_4 = -0.988332$
$x_5 = 0.855167$	$x_6 = -0.754975$
$x_7 = 0.676507$	$x_8 = -0.613218$

We show the trajectory (for $-7 \leq t \leq 100$), initial point, and these first eight crossing points in Figure 3.

5 Hamiltonian Systems

The energy function $E(x, y)$, used in both the Coulomb and quadratic damping examples above closely resembles a Hamiltonian in the sense that the energy function arises from a first integral of the system. There are two major differences however as (1) the Hamiltonian is a twice differentiable function of two variables and (2) only applies to conservative systems, that is systems with no dissipation and in particular no jump discontinuities. This author would hope that the investigations above show that these ideas are readily integrated into a beginning course and can provide useful tools that will assist students in other investigations.

6 Conclusions

Students in beginning courses already know about kinetic energy ($\frac{1}{2}mv^2$), potential energy (a function of position) and Newton's Law as they routinely use these ideas in developing the linear spring and the undamped pendulum models. Coulomb and quadratic damping, arguably more realistic than the classical viscous damping, are easily introduced from an energy point of view. The energy approach permits us to see globally the behaviour of these models and we can compute the amplitudes of the oscillations effortlessly. Basin of attractions are also easy to determine, although space did not permit us to do so here. More details can be found in [8] and [9].

We are able to introduce new and deeper topics by employing readily available powerful software and hardware. And in doing so, we demonstrate a new way of thinking through the integration of theoretical, graphical, and numerical investigations. All three go hand-in-hand.

We conclude with some student research projects.

Student project 1. The logarithmic decrement (for example see Boyce and DiPrima [4]) is a useful concept for the linear model having viscous damping. Given a physical model, numerical estimation of the logarithmic decrement can be used to estimate the damping coefficient. Is there an equivalent concept for Coulomb and for quadratic damping that could be used to estimate the decay in amplitudes such damped equations?

Student project 2. Investigate the relationship between frequency and amplitude of the vibrations in both the damped harmonic oscillator and damped pendulum examples with respect to the initial conditions.

Student project 3. Investigate the oscillator energy equations by plotting energy $E(x)$ versus x for both models.

Student project 4. Investigate oscillator equations where the damping is more complex by assuming a combination of viscous and quadratic damping.

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Challenges and Responses: Lessons from Students in Mathematics Bridging Courses

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Mathematics underpins the study of many different disciplines at university, yet, in the past fifteen years, there has been an alarming reduction in the number of students taking higher level mathematics options in year 12 [1] and a corresponding increase in the number of students enrolling in mathematics bridging courses. Mathematics bridging courses present important opportunities to investigate challenges of teaching and learning as students are being inducted into the tertiary study of mathematics. In a previous paper, we reported on the challenges of teaching mathematics bridging courses [2]. In this paper, we investigate the other side of the coin — students' perspectives. We present qualitative data, collected through a series of structured e-interviews with students, on the challenges of learning mathematics in a bridging course. We further report on how the students responded to these challenges and their ideas about teaching and learning. Finally we discuss the lessons that can be learned, including how bridging students' diverse mathematical backgrounds and experiences can be a resource to facilitate learning.

Keywords: mathematics bridging courses; challenges; students' experiences

Introduction and Context

The challenges of teaching an increasingly diverse cohort in higher education are felt in every discipline but arguably none more than in mathematics units. Mathematics underpins many different topics in science, engineering and other disciplines. Further, the nature of mathematical knowledge presents particular challenges — as both the content and way of reasoning builds on students' previous knowledge and experiences.

In Australia, in the past fifteen years there has been an alarming reduction in the proportion of students taking higher level mathematics options in the final years of high school and a corresponding growth in proportion of students studying mathematics at a pre-calculus level [1]. Unfortunately for this latter group, their mathematics preparation is inadequate for their university studies.

University entry criteria may exacerbate the problem. In Australia, many universities do not have subject prerequisites for entry into degree programs including Economics, Science or Engineering. Varsavsky [3] observes that students with gaps in their mathematics background are not blocked from pursuing Bachelor of Science degrees and showed that preparedness of students entering higher education affects both success and engagement with university level mathematics. Consequently, the mathematical under-preparedness of commencing undergraduate students is an important issue for university teachers not only in mathematics itself but in many other disciplines.

In Australia and overseas, many universities offer mathematics bridging courses — short courses available before students commence their degree program — as one of the ways to help students prepare for their mathematics units and ameliorate students' difficulties with mathematics in their chosen degree programs [4,5]. Recent reviews [6] of the research into bridging mathematics in the Australasian region have indicated

consistent areas of investigation, including evaluation of specific courses, diagnostic tests and other ways of determining students' needs and overcoming mathematics anxiety. Yet there is little research about what bridging courses mean to students studying them.

Mathematics bridging courses present valuable opportunities to investigate challenges in teaching and learning mathematics as students are being inducted into university study of the discipline. Mathematics bridging students are diverse in educational and mathematical background, intended degree program, confidence and willingness to engage with mathematics and expectations about their study. Hence the bridging environment represents a microcosm of the diversity that is a hallmark of introductory mathematics classes in higher education.

The literature indicates that teachers in higher education use a wide range of strategies to recognise and accommodate students from a range of social and educational backgrounds. These include small group activities to help students get to know one another and feel a sense of belonging [7], explicitly drawing on group (cultural) and individual differences in class discussions and activities [8], and providing new students with equitable access to relevant institutional and faculty information in a portable CD [9]. Support for new students is particularly important in view of the widening participation of 'new' demographic groups in higher education [10] and extent to which self-motivation and independent learning are required for university, as this can be problematic for new students [11].

A theoretical framework advanced by Barnett [12] p. 26, proposes that sustaining and developing a student's potentially fragile "will to learn" is of crucial importance "ahead of both knowledge and skills", and that anxiety could be a key element of losing the will to learn. Lowe and Cook [13] p. 75, found that the most challenging time for learners was at entry to higher education, and stress that proactive strategies and academic support should reach new students "before they have an opportunity to experience feelings of fear, failure, disappointment and confusion". Hultberg *et al.* [14] add that a well-planned and stimulating introduction to higher education is a key part of the transition process.

The transition from school to tertiary study of mathematics has been identified as an area of particular and growing research interest. Hong *et al.* [15] report a mismatch in the perspectives of secondary school mathematics teachers and lecturers about transition issues in mathematics education, particularly regarding the importance of calculus — included in most mathematics bridging courses. Leviatan [16] refers to a 'cultural gap' between school mathematics and tertiary mathematics with tertiary mathematics involving more abstract concepts and formal proofs.

Our report is set in the context of a project investigating the experiences of teachers, coordinators and students in mathematics bridging courses. We define bridging courses as preparatory courses that are intensive, 40 hours or less of instruction, held in late January or early February. In a previous publication [2] we have written about responses of teachers and coordinators in answer to the question: What are the most important challenges you have encountered in teaching and/or developing a mathematics bridging course? We found that four major themes emerged from those findings:

- challenges of teaching a diverse student group in a bridging course format
- challenges of teaching complex mathematical concepts
- changing students' perceptions about mathematics and themselves as learners
- organisational and logistic challenges (in the case of coordinators of mathematics bridging courses).

In this paper we investigate the other side of the teaching/learning coin to consider

what we can learn from students' perspectives. Our two primary research questions are the following. What are the challenges experienced by students in mathematics bridging courses? What do students report helps them manage these challenges and learn effectively? We then ask: what is the match between challenges as expressed by teachers and students and what are the implications for teaching and learning mathematics bridging courses?

Methodology

The project draws on qualitative methodology developed by Gordon and colleagues [17, 18] that collects data by means of asynchronous email-interviews with at most three returns. In these e-interviews the original set of interview questions are open-ended and designed to be specific to the respective respondent group, with the second and third interviews tailored to probe, in depth, participants' responses from earlier rounds.

Students studying a mathematics bridging courses at the authors' university in 2010 were invited to take part in the qualitative research. Students were enrolled in one of the two levels of bridging courses available: intermediate (called 2 Unit) or advanced (Extension 1, also called 3 Unit level). Each course included 24 hours of face-to-face tuition. The courses were taught in classes with a maximum of sixteen students; a total of 19 classes.

Fifteen students completed the full round of at least two e-interviews before our cut-off date, resulting in rich data of over 20,000 words (questions and responses) and it is their responses we report here. Participants gave informed consent and we quote short extracts from the students' transcripts under pseudonyms chosen by the students themselves, in line with our ethics approval.

Our participant students were a diverse group of nine males and six females. Seven students were recent school leavers while eight were mature aged students with seven students aged thirty or more. All but two students were enrolled in tertiary study, with eleven students commencing an undergraduate degree program. Two students had commenced their degree programs: one was a postgraduate student. Four students had previously completed undergraduate degrees. Six students were enrolled in the intermediate mathematics bridging course (2 unit) with nine in the advanced mathematics bridging course (Extension 1).

The first email included a welcome message and a set of initial questions. Question 1 asked for background information such as the student's degree program, intended unit of study in the coming semester, gender, age and mathematics background. The remaining questions were deliberately open ended with the intention of enabling students to explore and articulate aspects of their learning. These questions included the following. *What did you expect to achieve by studying this maths bridging course? What is a good maths bridging course? What is good teaching in a maths bridging courses? What were the most important challenges you encountered in the maths bridging course?*

Follow-up email interviews probed and ask for clarification and amplification of respondents' initial answers. Some examples are: *Were there any teaching approaches your tutor used that helped to address this problem of different levels in the class? What advice would you give to a friend or colleague who was preparing to start a maths bridging course?*

We focus primarily on students' responses to the following question and follow-up discussions: *What were the most important challenges you encountered in the maths bridging course?* However, each transcript was considered in full to ensure accurate accounts of students' perceptions and ideas about teaching and learning. We do not

make any claims as to representativeness of the sample, nor is it our intention to categorise participants. Rather we investigated in depth the themes that emerged from the students' responses.

Findings

We have separated the themes analytically and use students' own words (shown in italics) to illustrate them. Most students referred to several of these themes in their e-interviews.

Challenges of learning mathematical concepts

This was a major theme among almost all the students.

For some, the challenge began with the mathematical notation. Lemon reported: *Getting used to new notations was a challenge at first. When being taught something new, and having to understand what was being written on the board, it was a bit hard to follow.* Susan likened this to *staring at something that looks like a foreign language*. For Monkey, this situation was exacerbated by the fact that *I couldn't see any practical application for all these formulas*.

Meeting new mathematical concepts for the first time was daunting for most of the students. As Joey commented: *studying topics that were not taught in 2 U [intermediate] maths of HSC was challenging*.

Students experienced considerable difficulty with the more conceptually complex topics, as was to be expected. As John noted: *attempting to understand the new/more difficult material would be a challenge, which is always the case for anybody attempting to study new/more difficult maths*. Carly found that the later topics were more difficult for her to work through and that *it was easy to get frustrated when I couldn't grasp concepts as quick as everyone else*.

Several students articulated their approaches to addressing the challenges and learning new concepts, which included working both during and after class.

Volvo reported: *I would need to keep asking the tutor questions and re-reading the red booklet course notes to help me get through my challenges during class time*. Matt reported that: *content wise, I could mostly follow what the lecturer had to say if I'd read up first and revised the next day (ie it took several goes before I understood the chapter)*. He found that some of the exercises seemed *a big step up from the chapter*.

PJ felt that he needed to practise *problem solving and careful calculation*. Kannen added: *and algebraic manipulation*. Aura2 described how he was: *gearing my mind to the challenge by buying 2 unit and 3 unit maths, sitting at the table reading and practising the exercises*.

In summary, this challenge related to cognitive aspects of learning mathematical concepts and highlighted students' experiences and difficulties in trying to assimilate new and complex ideas.

Challenges related to the students' personal stories

Students' personal stories have a bearing on the challenges they perceived in the mathematics bridging courses, and these ranged from their previous experiences learning mathematics to their current situations.

Several students were mature age students who were returning to their studies following a break from school. For some, like Monkey, the lack of familiarity with mathematical notation resulted from: *not having brushed up on the previous learnings from year 10 (12 years ago)*. Kannen was particularly challenged by *having to use*

modern calculators while Matt acknowledged that: coming from an employment background ... meant the material was not as familiar as it would be to a recent school leaver, so I had to work harder to understand what I'd forgotten.

Some students reported challenges of a more fundamental nature. Susan, a mature aged student, reported that a major challenge was overcoming her fear of doing mathematics: *because I have never really understood it and thus I believe[d] I cannot do it.* She emphasised that it was important to her to really understand what she was doing: *so if faced with a problem that was a little different to what I had learnt I was still able to find a solution using the knowledge I had.*

Kannen highlighted another challenge common to those who were holding down a job while studying — *finding time to do the exercises.* This was an issue for both Matt and Hagrid. Hagrid acknowledged that it was a challenge to find *time to do the homework, which I should have done but didn't. ... I did not have a great deal of discretionary time after class.* Matt's solution was one of *time management — at work I used to book a small meeting room at lunchtime for 4 days a week so I could concentrate on last night's homework and read up on today's subject.*

Hence, in this category, students' reports expressed affective and situational factors that impacted on their learning in the mathematics bridging courses.

Challenges of learning in a 'university way'

For many students the expectation that they would *work independently* [Joey] during the bridging course was the greatest challenge and for some, like John, this issue of *attempting to complete all of the work* was exacerbated by the condensed timeframe of the course.

Boris Grishenko expanded. *We really had to understand a lot of the material ourselves and work through ourselves given the limited time of the bridging course.* He recognised the value and importance of working independently rather than being 'spoon fed'. *Personally I find that kind of learning more beneficial to me than having a teacher spew out a bunch of stuff you have to copy down, like they did in High School.*

Independent learning was the 'only' way of pinpointing her weaknesses, realised Volvo and thence: *work on them.* The bridging course acted as a "test" to see if she should pursue her career goal of teaching mathematics. She believed that since the bridging course was a very demanding course: *if I am able to handle the workload, and complete the course successfully I know that I should further pursue teaching mathematics.*

For some students adapting to *doing maths the 'uni' way*, as Kannen put it, was demanding and unfamiliar at first. He believed that bridging course teachers should *emphasise that the more the student works at it, the easier transition will be to study math at uni.* Lemon had some further advice on this point. *Understand our background. Most of us were high school students or mature age students so we hadn't studied for a long time and were used to being taught the high school way.*

This category highlights that students studying mathematics bridging courses are adjusting to change — making the transition from secondary to tertiary education.

Responding to challenges — students' ideas on teaching and learning

Many students recognised that teachers in mathematics bridging courses had to manage teaching students of all levels. As Monkey summarised: *In my bridging class there were many students straight out of school, but also many mature students who had not done any mathematics for many years.* Bea Arthur commented that: *it was*

surprising and funny to be back in a room full of seventeen year olds! I think they called the teacher 'Sir'.

To address this diversity, good teaching appeared to be an essential ingredient of successful learning in the bridging course. Initially, Susan had an unfortunate experience with a tutor she did not understand. *When asked during my first weeks of this course I said nothing had changed, math was still awful and I still did not understand.* She joined a different class for the third week *and it was a breath of fresh air. The difference was noticeable from the moment I walked in.*

To Matt a good teacher was empathetic: *a knowledgeable and keen tutor (who remembers what it was like to be us).* He appreciated personal attention very much: *especially when I think that a normal uni lecture will have some hundreds of students in one big lecture hall.* Aura2 agreed that: *direct one on one contact was helpful as everyone has a different way of learning.*

A teacher who *allows the course to be casual, and is patient/accepting of each person's strengths/weaknesses/ways of learning* was best for mathematics courses in Carly's opinion. She illustrated how her tutor: *showed numerous ways to answer the same question so as to be flexible with different people's learning styles. He explained many concepts using analogies, for example with functions he used the idea of a machine taking in "inputs" and pumping out "outputs".* In addition, Bea Arthur suggested that it helped if the tutor *explained the topic clearly and patiently, without presuming too much prior knowledge. Not using unnecessarily confusing terms is good.* She added that: *going through examples together also helps to keep the whole class at a similar rate of progress.*

A pedagogical skill especially important for teaching a mathematics bridging course was *an ability to be concise in describing the material*, specified Hagrid. He felt that the tutor needed to: *gain a good idea of what the students know and don't know so that time can be managed properly, e.g. less time on things which students know or can understand easily and more time on entirely new content.* Kannen's criteria for good teaching were: *explaining a deeper understanding rather than an operational process; explaining the context in which the mathematics is usually applied and letting students know that they will have to work hard on maths in first year.*

Working in groups is a teaching strategy that is built into the structure of the bridging courses. Many students commented on this strategy. Joey summarised: *I found that doing examples of questions on a topic as a group was the best teaching method. Also being able to: bounce ideas and knowledge off each other made it more exciting.* Matt agreed that the class interaction is good, explaining: *if you don't quite follow a line of argument, and if you don't ask then someone else will (you're less inhibited in a small class than in a mainstream lecture hall).*

Carly explained how this group strategy worked for her: *Often, the tutor's explanation/examples were clear to only one person in the group, and so they were able to "teach" the others in the group who hadn't completely understood it.* In addition, Carly found that this interaction allowed friendships to form: *which led to a more relaxed tutorial session, and so we were able to work more efficiently in helping out each other in each of our weaknesses.* Kannen elaborated: *If I was helping others, giving an explanation reinforced my understanding or showed that I didn't understand correctly. If others helped me, they sometimes also showed me a different perspective.* Boris Grishenko noted that: *this was great because it made up for the times where the teacher was helping someone else and was unavailable to help us.*

Not all students were positive about group work. Bea Arthur noted: *I was coming to class every day after work and was pretty tired, and also I really couldn't spend any*

extra time on the exercises outside of class, so to be honest I just wanted to get through as much as I could without having to regularly stop and explain how I got an answer. Susan, on the other hand, found working with fellow students self-affirming. I felt as though I had something to offer the students I was working with — not just leeching off their understanding or prior learning. I also found that I was able to offer them some help — on the occasions I seemed to get it when they didn't.

Various other strategies to ameliorate difficulties were suggested by the students. Monkey thought that: *grouping students by knowledge level [would help] to make the classes flow smoother and ensure that the classes don't go too fast for some students and too slow for others.* Lemon would have liked to be told to study ahead of lectures, saying: *understanding how to do something before being taught it in lectures is much easier than trying to keep up in lectures whilst still trying to get your head around the concept.*

From these reports we see that participants identified that good teaching was central to their learning and that empathy, pedagogical knowledge and a flexible approach were critical aspects of good teaching in a mathematics bridging course. Students also suggested ways to enhance the learning process — a key strategy for most being peer interaction.

Discussion

In this project, both teachers [2] and students recognised that mathematical concepts, particularly new ones, are difficult for students to understand and that these difficulties are exacerbated by the short time frame of mathematics bridging courses. Further, both groups acknowledged that the diversity of the bridging course students in terms of mathematical background, age, attitude to learning mathematics, expectations and personal situations make for a varied and demanding teaching and learning environment. Several of our teacher participants [2] recognised the potential difficulty of their students in adjusting to university, commenting particularly on the need for their students to develop autonomous study skills and take responsibility for their own learning. In accord, the students' views reported in this paper show that the shift from secondary school to university is a great challenge with considerable adjustment needed to manage time, priorities and, perhaps most importantly, to learn independently.

Students enrolled in bridging courses are in transition and their responses indicate that the students were both looking back and looking forward. Looking back, students acknowledged their previous difficulties with mathematics not only on the cognitive level but also affectively — and recognised the opportunities afforded by the courses to ameliorate these difficulties. More proactively, students showed awareness that the bridging courses provided the opportunity to increase confidence and enable self-development: to transform their futures.

Our findings show that the mathematics bridging courses helped students connect with mathematics, in some cases after a long period away from study. The bridging course changed some students' perceptions of themselves as learners of mathematics. Susan overcame her fear of doing mathematics; a challenge identified by teachers for a significant number of students, and described by one as “the hardest challenge” [2], p.38.

Data about experiences of mathematics bridging courses could inform advice given by teachers and careers advisors at secondary school and influence decision-making by students, perhaps discouraging at least some students from taking the easier level of mathematics at senior levels of secondary school. We argue, too, that reflection and discussion is needed about teaching complex concepts at the appropriate levels. A

question raised by our research is: Are there effective teaching approaches for bridging mathematics that are distinct from those used in junior mathematics courses? This in turn relates to a question, raised by Taylor and Galligan [19] p. 10, who ask: “What constitutes success for students in mathematics bridging courses?”

Conclusion

This project contributes to the debate around early support and orientation of mathematics students at university by giving mathematics bridging students a voice in the discussion. The experiences reported by our respondents highlight that bridging course students are not only tackling the challenges of studying mathematics in higher education but are also participating in the socialisation process of “becoming university students” [20], p. 351. Mathematics bridging courses provide an opportunity for student development — promoting student confidence, encouraging independent study and providing an environment where students can learn to work with others. The process of transition to university life could begin in bridging courses.

Teachers of mathematics bridging courses can learn from students’ experiences to develop their approaches to teaching and to take account of diversity in their classes — *to be flexible with different people’s learning styles* [Carly]. Teaching strategies suggested by the students include making overt ideas on how to learn mathematics and explaining the context or application of the mathematics. The students’ reports show that diversity can be a resource; the different strengths and backgrounds of the students that emerged during peer group interactions helped ameliorate *each of our weaknesses* [Carly] and provided *different perspectives* [Kannen].

Finally, mathematics bridging courses provide a significant opportunity to promote students’ receptiveness to learning mathematics before there are, as Matt pointed out, *hundreds of students in one big lecture hall*. He concluded about teaching a mathematics bridging course: *Gotta be one of the more pleasurable teaching jobs going around. (Students) are there because they *want* to be there*.

Students’ expressions of motivation resonate with Barnett’s [12] proposal that the will to learn is the basis of students’ academic persistence in the face of uncertainty and difficulty. Nurturing this will to learn in bridging courses could contribute, vitally, to students’ engagement with their ongoing study of mathematics.

We need to continue to monitor how students are experiencing mathematics bridging courses and how these courses impact on the critical issues of students’ performance and retention in first year mathematics — a topic of our ongoing research.

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Towards an Understanding of Statistical Task Design

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This paper describes a rich *sampling* activity that was the original creation of Professor Nye John and Dr David Whittaker from the University of Waikato at the end of the last century. Slight modifications have been made to the original conception in order to provide a stronger experiential foundation and greater possibilities for learning. The primary modifications have been the addition of a ‘fun’ element (sampling pieces of chocolate –rather than wooden blocks- and rewarding the student who produces the *best* estimate with chocolate), the development of extension activities which use the data produced in the workshop (important statistical concepts encountered during the semester are mapped back to the sampling activity) and a website to assist with data collection and recording. The activity has been named (whimsically) “The Chocs and Blocks” activity – *chocs* – because the task is to estimate the average weight of a piece of chocolate using different sampling methods; and *blocks* – because the original activity was structured around blocks of wood, rather than pieces of chocolate. Presented here are my reflections on this activity as a meaningful statistical learning task. I begin by briefly describing the task (with my *points-of-noticing* appearing in italics) and then continue with discussion of these *points-of-noticing*, focusing on their relevance to the design of statistical tasks. This discussion is grounded in my understandings of teaching and learning that are informed by the notion of embodied knowing (Maturana and Varela, 1987; Varela, Thompson and Rosch, 1991), and ideas from complexity science (Davis and Sumara, 2006; Davis, Sumara and Simmt, 2003; Davis and Sumara, 2003; Pirie and Kieren, 1994). I also draw on criteria for good learning activities suggested by Ahmed(1987), understandings of mathematical task design as proposed by Mason and Johnston-Wilder (2004), and research into creativity in higher education (Jackson and Sinclair, 1981). I conclude by offering some suggested points to consider when designing and developing meaningful statistical learning tasks.

Some time ago an activity entitled “Chocs and Blocks” was incorporated into one of the statistics service subjects provided by our department. The subject, Experimental Design and Data Analysis (CHK name), was presented to students from the Faculty of Land and Food Resources as part of their degree program.

Our intention was to use the activity as an ice-breaker and grab the attention of our reluctant cohort with a fun statistics activity that also had benefits for learning. More specifically, we wanted to

- provide students with a memorable experience that teaching and learning staff could draw upon throughout the semester when visiting some of the ‘big ideas’ in statistics such as variability, sampling, randomness, estimation, bias, precision, independence, blocking (amongst other design principles).
- set the tone for classroom practices for the rest of semester

Subsequently, the task was run in several different statistics courses and also at different stages in the course. In each instance, the student cohort (and lecturers and tutors) varied, as did the didactic relationship between teaching staff and students that is, the *milieu* (Brousseau, 1997). Even so, there seemed to be a consensus from teaching staff that the activity was a ‘good one’. But what this meant was not made explicit.

By looking into back into (re-searching) this task I hoped to come to a greater understanding of what made this task a ‘good one’. In particular, I was hoping to identify the possibilities, embedded within the task, for encouraging (statistical) knowing, and the opportunities afforded by the task for challenging and building upon student’s existing statistical knowledge structures (what I refer to as their *statistical knowing*).

I begin by describing the task, with my reflections (points-of noticing) appearing in italics at various stages of the description. I then move into a more general discussion of my reflections (points-of-noticing), focusing on their relevance to statistical task design.

Describing the Task

The chocs and blocks task is a variation on an activity that I first encountered while teaching in the statistics department at Waikato University. The designers of the original task were Professor Nye John and Dr David Whittaker (John & Whittaker, 2001) and it was incorporated into an introductory statistics course for business/management students.

In the current version of the task, students are asked to come up with a strategy (that they can later implement in the class) for ‘estimating’ the average weight of a piece of chocolate (irregular wooden pieces in the original version) on a tray containing one hundred irregular broken pieces of chocolate.

Students are told they can solicit the tutors for information that may help in developing a strategy– but whether or not the tutors will provide that information is another matter! Tutors are instructed to encourage student’s to ask meaningful questions (questions that will help them to come up with a workable strategy).

It is important that tutors provide very little to no direction here. At this stage of the task the purpose is to encourage the innate abilities that students bring to class, particularly their ability to imagine and be inventive and creative. Therefore, it is important that tutors do not close the task down by imposing their expectations of the task upon the students.

By offering a tangible reward (chocolate) the intention is to appeal to students’ innate competitiveness- can they come up with a ‘better’ estimate than anyone else, is their strategy going to be ingenious?

Instructions to students: - No calculators or instruments to be used. If a student asks if they can have the weight of a specific piece/s of chocolate that they have chosen tutors can answer in the affirmative – but only provided the student has initiated the request! Maximum number of weights allowed is 10.

...at this stage of the task, the emphasis is on getting the students to think creatively about asking meaningful questions in order to obtain meaningful information (emphasis on ‘meaningful’). To experienced researchers this may sound trivial but over ten years of experience with this task has shown me that many students find this part of the activity very challenging ... and it takes some time!

Over the ten years that I have been running this activity I have noticed that students seem to have increasing difficulty with this stage. I am wondering if this has to do with a downward trend in creative intelligence (Kim, 2011) or if it is something to do with the amount or type of mathematical training students bring to the task. Without further research all I can say is that the groups that had no difficulty coming up with questions were the Year 11 (Further Mathematics) students (over 3 separate years, but same class teacher), students from our Land and Food Resources course (mature-age and minimal current mathematics) and students from the Statistics for Researchers course (mature-age researchers) run by the Statistical Consulting Center. I am intrigued. In my view,

the difficulty does reflect an inability to think creatively. Perhaps students of today are smarter than those of yesteryear but at the expense of creative intelligence, as Kim's research with Americans suggests. Further research is needed in order to shed light on this question

Some of the more common queries from students and associated tutor responses:

Student queries	Tutor's common response
What is the weight of the whole tray?	Can't say
How many pieces are on the tray?	100
Can I hold the tray	Sure, just don't drool on the chocolate 'cos we want to eat it afterwards
Where did the chocolate come from?	Haigh's
Can I weigh the tray (a piece) myself?	Do you have a set of scales handy?
Can I touch the chocolate?	Nope... OHS reasons
Is it all the same type of chocolate?	Yep... and it is yummy
Where did you buy the chocolate?	Haigh's
Was it in a block?	A broken sheet, rather than a block

Tutors circle the room paying attention to what strategies are emerging (often there are two tutorial groups 'ganged' together for the activity, so two tutors in the room). When they feel that everyone has arrived at some strategy or other, students are then invited to share their strategies with the class as whole¹. At this stage, students generally arrive at taking a sample of 10 pieces and calculating the average weight of the 10 pieces. However, it is not uncommon for some students to use a sample with fewer pieces than the maximum (e.g. selecting what they, presumably, consider to be a model value 'by eye').

The purpose here is to create the opportunity for students to test their belief that their method is appropriate for coming up with the 'best' estimate ... and open the way for further conjecture about the quality of the random sampling method.

Students are then referred to the ²website that has a simulation of the broken pieces. Students enter the ID numbers for their selected pieces, and then the simulation returns the individual weights and the mean weight for the selected sample. The site clearly indicates that a maximum of ten weights can be recorded

¹ There is a timing issue here – in more recent years, with students less capable of coming up with a sensible strategy (some of our Biomed students felt it was appropriate to guess, others thought they would estimate the average weight based on comparison with the known weight of a chocolate Freddo frog) it has seemed more productive to prompt the less creative students by encouraging everyone to openly share their strategies with the group. This is in the interests of moving the activity forward. In earlier presentations there has been no need to talk strategies at this stage – students were just set loose to determine their estimated mean (after being given the weights of their chosen pieces).

² With smaller groups it was possible to go to each student individually and provide them with the weights of their chosen pieces. Students selected their pieces from the simulation (the wooden blocks). Two benefits of this approach were, students could actually handle the blocks and play with them when deciding upon their sample; and secondly, the task remained open longer because students who had no strategy were not given any hint as to the maximum number of weights they were allowed.

...at this point it is important to stop the students from taking multiple samples! Even when they are told their estimate must be based on one sample some more creative students have tried to pool their samples so that they end up with the population of 100 weights. Re-directing them to the educational objective – that is, when it comes to selecting a representative sample, how good is human judgement? This is usually the first indication of the intended purpose of the task.

Students record their weights and their mean on a task sheet and also briefly note the strategy they used to select their sample.

...by 'putting pen to paper' students begin to take ownership of their strategy and are thus encouraged to reflect consciously upon their choice.

Whole group – students call out their individual sample mean (*students taking ownership – see below*) and results are presented on the board using a stem-and-leaf plot. Tutors lead a class discussion, drawing out the strategies particular students used (strategies associated with largest mean, smallest mean, most common mean). It is also meaningful at this point for the tutors to direct students to consider the assumptions made when adopting particular strategies.

...by having the students 'call out' their sample mean so the whole class can hear, students are encouraged to take greater ownership of their estimate.

...the majority of students go for what they think will be representative, many say they chose randomly when, with further probing from the tutor, it is revealed that they actually meant representative; on a few occasions students have chosen a systematic sample and provided good reason as to why they thought this was appropriate. Generally speaking, students are not even aware that they have made assumptions about the distribution of weights. Probe questions that have been used at this point (in order of increasing amounts of direction)

- “in coming up with this estimate what assumptions have you made?”
- “what assumptions have you made about the distribution of weights?”
- “what does this tell you about how the blocks were numbered?”
- “what did you assume about the relationship between the weights and the numbers of the pieces?”

At each stage of probing, it is appropriate to also ask the question, “how would you check this assumption?”

Notably, the vast majority assume there is no relationship between how the pieces are numbered and their weights and that the distribution of weights is symmetrical. Both of these assumptions are re-visited later in lectures.

...open discussion and sharing to provide opportunity for students to further reflect on their choice of strategy.

Discuss the shape of the distribution (of the estimates) and variability in estimates, middle 50% of estimates, mean of estimates.

Students are asked if they are happy with their estimate or if they would like to change their estimate based on what everyone else came up with

...surprisingly, it is extremely rare for a student to shift from their original estimate – even when they based it on a sample size of 1!

The true mean is now revealed ... and students are generally amazed that they were so wrong!!! The producer of the *best* estimate gets to choose a piece of chocolate from the tray.

If assumptions have already been discussed, now is a good time to re-direct students to consider the validity of their assumptions and how they could check their assumptions.

...teacher is now provoking and directing attention to processes involved in a statistical analysis – making assumptions and checking assumptions

To further encourage this part of the investigative cycle, the population of weights (and associated ID number for each weight) is made available (on the Learning Management System) for students to check assumptions for themselves using MINITAB. The checking of assumptions is followed up more formally in lectures at a later date.

...students are also informed that *overestimation* is common – it happens every time the task is run. Perhaps it is an aspect of human perception? The challenge is put out to investigate further and report back (this works well if there are some psychology students in the cohort)

The revelation that humans are not very good at choosing a representative sample and *that they consistently overestimate size, provides the ‘unexpected’ or ‘surprise’ element. Humans are predictable... and predictably wrong!* I interpret this ‘surprise’ element as equivalent to Piaget’s “perturbation”, and what a *complexivist may describe as “a disruption to the pre-existing knowledge structure” - in either case, it is creating the opportunity for learning to take place.*

The students are then invited to repeat the process but this time choosing a randomly generated sample. This part of the task is very quick... students seem to be eager to find out if the *random* sample mean is, on average, a better estimate.

The means from the random samples are then compared with those from the judgement samples.

...students are generally quite curious by this time to see how the estimates (means) fall. Great amazement that the random means are centered about the true population mean and with much less variability ...which is very nice groundwork for future discussions of sampling distributions!

Chocolate is distributed all around!!!

The full details of the activity, recording sheets and some probe questions are available, on request, from the author.

Discussion

In reflecting on this task my attention has been drawn to what I liked and disliked about the task and from this has emerged ... a clearer understanding of what directs my choice of tasks and specific things for me to think about when developing tasks. Two primary issues have become apparent to me: first, what I am drawn *to notice* about this task is necessarily informed by my view of the nature of knowledge and how students learn; and second, the over-arching question for me, in any task design is ‘what is it possible to learn given this task, in this context (note that *context* includes socio-cultural, psychological, historical *baggage* that students and teachers bring with them)?’ Which is akin to considering the purpose of the task.

My understanding of teaching and learning

I understand learning as an ongoing, recursive reorganizing of one’s knowledge structure (Davis & Simmt, 2003; Davis, Sumara and Simmt, 2003). Learning occurs in the interactions between the learner (knowledge-producer,) and their environment (which includes, amongst other things, learning tasks). The learner is viewed as an autopoietic system, embedded within larger systems (Maturana and Varela, 1987; Varela, Thompson and Rosch, 1991) and we are encouraged to think of the learner as a living and learning system bringing forth new knowledge structures (Davis and Sumara, 2006). In this perspective, learning is occasioned by experiences that are new,

unexpected, or in some way challenge the learner's existing knowledge structure (eg a contradiction or counter example). Such experiences/disturbances to the *knowing system* can be likened to Piaget's *cognitive dissonance*. Something disrupts the taken-for-granted or assumed position and so a new position must be established so that the learner retains their integrity.

The knowledge structure that students hold to is the one that produces the *best fit* to their personal experiences-to-date. Provided the student is open to learning, this knowledge structure *shifts* (ie learning takes pace) with any perturbation/disturbance in the system, such as a contradiction or counter example, or new experiences that challenge the structure to be re-organized: new (additional) connections are made (corresponding to greater depth of understanding on the part of the student), (some) old connections may be broken (no longer providing 'good fit' to the learner's experiences).

In this view of learning, we cannot predict what knowledge structures will emerge (pearls come in a variety of shapes sizes and hues), but we can provide experiences that continually challenge existing knowledge structures so that students stay open to learning.

In terms of statistical task design this suggests that the activity should challenge the student's existing perceptions and statistical understandings, and have an element of surprise.

This still does not guarantee that students will learn what we want them to learn. All we can truly aim for is to encourage them, via the processes embedded within the task, to be aware of and make connections between what we consider to be the important statistical concepts associated with the task.

The extent to which this is successful will depend, to some extent, upon how the task is incorporated into the curriculum which includes using the data produced from the activity in later topics, making explicit the statistical concepts embedded within the task when covering related statistical concepts, and enriching the task through extension activities embedded within later topics. All the while, referring back to students' initial experiences (with the chocolate). This curriculum process resonates with Pirie and Kieren's (1995) notion of learning as a recursive, adaptive, spiral-like process. Any ongoing dialogue surrounding the task would (ideally) encourage reflection and focus on making explicit the links between the important statistical concepts and also between associated statistical procedures.

A further point arises here and that is the importance of keeping the task open for as long as possible. A delicate balance needs to be maintained between providing too much instruction and allowing students a completely free-reign in coming to know. A problem that we have encountered in the past is that some tutors, when they become overly familiar with a task, have a tendency to provide too much direction. Essentially, they impose their understanding of the purpose of the task on the group. Students are then unable to develop their own knowing, they adopt the knowing of the tutor. Students end up being able to copy, but they can't think creatively about a task!

Many of the attributes that Jackson and Sinclair (2006) associate with creative thinking resonate with what I refer to as a *living and learning system* (Gunn, 2002) - that is, a student who is developing their own knowing, generating unique knowledge structures. For example, Jackson and Sinclair describe a creative thinker as being mentally mobile and able to change perspectives, as having a high tolerance for complexity, disorganisation and messiness, skill in critical thinking thereby enabling ideas to be evaluated, and comfortable with discovery modes of being. This type of thinking (I believe) is important for students across the board and so it needs to be considered when reflecting on statistical task design.

Purpose of the task

I imagine that most educators consider that the purpose of any learning task is to stimulate the process of learning. Working from my understandings of the nature of knowledge and the way students learn, any learning task becomes about ‘educating awareness’ and hence increasing possibilities for action on the part of learners. Mason (2004) captures this essence in his discussion of the design of mathematical tasks (I have modified his comments to the *statistics* context):

“The purpose of a task is to initiate *statistically* fruitful activity that leads to transformation in what learners are sensitised to notice and competent to carry out.” (Mason, Johnston-Wilder, 2004)

I understand *statistically fruitful* activity to mean activity that engages students with the big ideas in statistics (such as variability, sources of variability, distribution, describing distributions, measures of variability and ‘centre’, modelling, sampling, sampling variability, estimating, predicting in the face of uncertainty, hypothesizing and testing hypotheses). Furthermore, I consider the activity is statistically fruitful if students are encouraged to think and reason statistically (Pfannkuch and Wild 1998b) which necessarily involves mapping connections between the big ideas in statistics and relating this ‘map’ to the lived world of experience.

Of particular interest to me are tasks that create opportunities for students to *engage in* statistical activity and *with* statistical concepts, to make connections to other topics and techniques and allows for multiple approaches, interpretations and re-presentations.

For me, possible focusing questions when interrogating a task can be broadly grouped into three (inter-related) areas: first, those that enquire about the statistical concepts embedded in the task; second, those that ask about the type of reasoning and thinking encouraged by the task and third, those that interrogate the opportunities for learning to take place. I illustrate, using my reflections on the Chocs and Blocks task.

What statistically fruitful activities are going on in this activity? What ‘big’ ideas in statistics are students being engaged with?

- variability, sources of variability, measures of variability
- distribution, describing distributions, graphical displays and summary measures for distributions
- statistical modelling, assumptions and checking assumptions
- sampling, sampling variability, sampling distributions

What types of thinking and reasoning does the task encourage?

- estimating, predicting
- hypothesizing and testing hypotheses
- statistical analysis process – reasoning regarding validity of assumptions
- statistical thinking³ and reasoning⁴

³ I understand Statistical thinking as a way of seeing the world. In this world view, data is dependent upon its context for meaning, variation is inherent in all things, modelling is a way of understanding the questions, and all predictions involve a level of uncertainty (which can be modelled and is measurable). So, for example, a statistical thinker would observe a difference in sample means but ask, “Is this a meaningful difference when considered against the background variation?” Statistical thinking involves being able to dwell in a world of abstractions ... whilst your feet are firmly planted in the world of lived experience.

- from yummy chocolate (sensory delight, embodied context) to sampling distribution theory (abstraction – concept)

What opportunities are provided to encourage new knowledge structures? What are the opportunities for transformation?

- disruptions to existing knowledge structures, the unexpected, counter-examples, surprise elements
- how is the task structured in terms of the balance between doing and theorizing, activity and reflection, the specific and the general
- what variations are in the task and what do they offer (extension activities and relevance)
- what connections are made explicit
- what opportunities are provided for re-visiting the experience
- how is the balance between creative exploration and directed instruction mediated

A further consideration is possible constraints upon the learning system

- Expertise of teaching staff (depth of statistical knowledge, educational awareness). That is, do tutors have sufficient knowledge of the course and subject to be able to ‘notice’ and act appropriately ‘in the moment’- ie to seize the moment
- Relationship that is established between teaching staff and students – is open dialogue possible?
- Timing of activity in the course – the re-visiting of the activity is an essential component of the task, enriching the task by mapping onto other statistical topics is important
- Will tutors *close the task down* and thus discourage the possibility of learning? This has happened in the past.

Most of which are beyond the control of mere mortals such as I, but still need to be considered.

In Conclusion

Interrogating this task has been a fruitful exercise for me. I now have some potential focusing questions that I can use in future task design. My view of the nature of knowledge and how students learn informs many of my choices. As does my understanding of statistical knowing – which includes statistical thinking and reasoning. Underpinning both of these is my commitment to the view of teaching statistics as *enculturation into a community of practice*. Therefore, any task that enables this enculturation is, to me, potentially fruitful. So, developing ways of seeing the world that are consistent with a statistical world view, and ways of practicing statistics that are consistent with the ways that professional statisticians work need to be embedded in tasks that I design for subjects that I teach. This relates to the meta-purpose (for want of a better word) of the task.

My reflections on this task have also brought to me some unanswered questions. I had always had the view that a task should be statistically fruitful, in that it should map

⁴ “Statistical reasoning may be defined as the way people reason with statistical ideas and make sense of statistical information” Robert delMas (accessible from the ARTIST website)

directly onto the course content and encourage a statistical worldview. What I had not explicitly considered was the importance or relevance of creative thinking. Consideration of this was provoked by the changing response, over time, of students to the very beginning of the task – where they have to come up with meaningful questions. Why some students find this such a challenge, and why increasing numbers appear to find it a challenge is still unclear to me. Both questions point the way towards some future research.

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Incorporating Partially Completed Worked Examples With Scaffolding Instructions in a Calculus Course to Facilitate Student Learning: To What Effect?

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The teaching of calculus in undergraduate courses has its share of curricular and pedagogical challenges. Suggestions on how to teach it well abound. To facilitate the learning of calculus of students enrolled on teacher education programmes, the authors devised an instructional strategy called guided-exercise in the form of a series of partially completed worked examples with instructional scaffoldings which the students used alongside the lectures, with assistance from the lecturers when necessary. This paper describes the rationale, design and use of these guided-exercises as an instructional strategy to facilitate and enrich student learning. It reports the students' perception on the effect the strategy has on their learning. Implications are discussed also.

Keywords: scaffolding, worked example, guided exercise, undergraduate mathematics, calculus

Introduction

How can we teach calculus better? The teaching of calculus in undergraduate courses continues to face numerous curricular and pedagogical challenges. In this paper, we report on an instructional strategy which we devised with the aim of helping students enrolled on teacher education programmes to learn undergraduate calculus. These prospective mathematics teachers have successfully completed and have passed Mathematics in the Singapore-Cambridge GCE A-Level Examinations.

Calculus II is one of the core modules for Year 2 student teachers (hence forth referred to as students) who are pursuing a Bachelor of Science (Education) or a Bachelor of Arts (Education) Programme to teach Mathematics in Singapore schools. It is a one-semester 12-week course comprising four topics : (I) Sequences, (II) Infinite Series, (III) Partial Derivatives and (IV) Multiple Integrals. Calculus II is delivered via two 1-hour lecture periods and one 1-hour tutorial period per week for 12 weeks. In Year 1, the students took Calculus I, which covered the topics: Functions and Graphs, Limits and Continuity, Differentiation and Its Applications, and Integrations and Its Applications.

Given that there is only one hour tutorial time for Calculus II per week, it was not always possible to carry out extensive scaffolding instructions to engage students in the conceptual discourse and help them consolidate and deepen their understanding of the concept taught in a typical tutorial session. A study by Lim, Ahuja and Lee (1999) in Singapore revealed that our pre-university students generally had a positive attitude towards learning mathematics, albeit a little too reliant on memorizing formulae or mathematical procedures in their approach to learning the subject. Their study also revealed the difficulties a number of our student teachers had in keeping up with the mathematical demands of the undergraduate mathematics courses. Indeed, over the years of teaching the same course, the authors also noticed an increasing number of Calculus II students showed difficulties and limited mathematical cognitive resources or

motivation to solve tutorial problems. In view of these students' weakness in mathematics in general and in a topic like Calculus in particular, our latest pedagogical innovation is an instructional strategy which we called 'guided-exercise'. These guided-exercises consist of a few partially worked-out examples incorporating conceptual and heuristic scaffoldings with the aim of guiding students to solve the problems independently, thereby strengthening their conceptual and procedural knowledge that they have learnt during the lectures. Students were strongly advised to attempt these questions before they solved the weekly tutorial questions which the students are expected to present in class.

Literature Review

We have deliberately been eclectic in our search for ways to teach better. To this end, we have borrowed ideas from the field of educational psychology. Cognitive load theory, as reported in Sweller (1988, 2010) and van Gog, Paas and Sweller (2010) suggest that integrating worked examples in a learning environment or as an instructional technique reduces the cognitive load on the working memory of the learners (especially for the novices), and allows them to focus their limited available capacity on studying the solution process. As a result, effective learning takes place.

The use of fully worked-out examples to teach mathematical concepts and procedures has been the predominant instructional technique in mathematics classrooms. Worked examples are also ubiquitous in many mathematics textbooks. There is abundant empirical evidence showing that learning from worked examples is more effective than learning from solving problems alone (Witter & Renkl, 2010). However, there are empirical evidence to suggest that learning from complete worked examples may not be as effective as learning from incomplete worked examples in which "blanks" are inserted in the solution steps of the examples (Renkl, Atkinson, Maier, & Staley, 2002). These "blanks" would force the learners to determine the next solution step on their own, helping them to acquire cognitive skills. Atkinson and Renkl (2007) concluded that "interactive" elements in the form of gaps (or blanks), prompts, and help on demand (in the case of computer-based learning environment), incorporated in worked examples can foster learning, in particular for learners with weak prior knowledge.

Apart from using complete or incomplete worked examples, mathematics educators also use scaffolding instructions to facilitate and guide students' in their learning. The concept of scaffolding as a teaching strategy originates from Vygotsky's (1978) socio-cultural theory and his idea of the zone of proximal development (ZPD). Student learning is posited to occur within the ZPD when the learner aided or scaffold by a "more knowledgeable" person. There are studies advocating the positive effect of scaffolding and recommending its use in instruction at different academic levels. For example, Johnson and Koedinger (2005) concluded from their study that scaffolding conceptual, contextual and procedural knowledge are promising tools for improving student learning while Valkenburge (2010) opined that scaffolding when used during tutorial sessions is a powerful tool for helping students to actively engage in their work and in promoting self-sufficiency. Witter and Renkl (2010), in a meta-analytic review of 21 experimental studies on instructional explanations in example-based learning, found that there was corroborating evidence to conclude that the instructional explanations had pronounced impact on learning.

Studies on using scaffolding instruction in partially worked-out examples have not been reported in Singapore. Hilbert, Renkl, Kessler and Reiss (2008) described the use of "heuristic examples"—a specially crafted worked-out examples with instructional

support in the form of self-explanation prompts—to teach proofs in geometry. Their study showed that learning with heuristic examples led to an improvement in students' ability to construct proofs and their understanding of proofs.

In our search for better instructional strategies to undergraduate calculus, we think it is valuable and novel to look at how scaffolding can be used with incomplete worked examples, which we called 'guided-exercises' to improve students' learning and to meet our instructional goal which is to equip our students with the most important concept(s) in each chapter and to enable them to link procedural knowledge with conceptual knowledge so that there is relational understanding (Skemp, 1976).

Design of the Guided-Exercise

In the building industry, scaffoldings are necessary and essential temporary physical structures that provide workers support to access and build another storey of a building. In the education setting, analogously speaking, the construction of knowledge and understanding by a learner requires cognitive scaffoldings to enable the learner to overcome hurdles to their understanding and knowledge acquisition. The idea of instructional scaffoldings has a similar function—for the teacher or knowledgeable others to provide support structures so that novice learners may advance to a higher level of learning.

According to Hartman (2002), the goal of scaffolding teaching strategies such as using models, cues, prompts, hints, partial solutions and think-aloud modelling is for students to become an independent and self-regulating learners and problem solvers. Holton and Clark (2006) argue that scaffolding students' learning does not require the educators to be face-to-face with the students in a classroom setting. It can be done in a form of specially prepared teaching instructions provided in written form and other means. They identify two types of scaffolding: one being the conceptual scaffolding which aims at promoting conceptual development of the learners and the other being the heuristic scaffolding which develops the skills of using heuristics for learning and problem solving.

Atkinson, Derry, Renkl, and Wortham, (2000) suggested that for worked examples to be instructionally effective, teachers have to look into three factors: (a) intra-example feature: how examples are designed and their solutions are presented (b) inter-example features: how multiple examples are sequenced and related to practice problems (in our case, tutorial questions) (c) how example-processing is done in the form of "self-explaining" by individual student.

Bearing in mind these factors, the need to scaffold students' learning in our intention of reducing students' cognitive load, and eliciting students' thoughts and explanations in order to achieve our ultimate goal of enabling students to acquire both the conceptual and procedural understanding, we designed the guided-exercise with the following considerations : (I) using instructional explanations; (II) crafting appropriate questions to engage students; and (III) incorporating controlled variability in the partial worked-examples. The focus of this study is not to test the cognitive load theory; rather it is to use the theory to inform the design of the guided exercises.

(I) Instructional explanations

Giving appropriate amount of instructional guidance and explanations is one aspect of the intra-example features that we need to deal with. We believe that students must involve or get inducted in solving the problem in order to make progress in their learning. Once a student is able to get started solving a problem, other scaffolding steps

become operative which would help the student make further progress in working out the solution. In this respect, initial instructional explanations can help bridge the gaps in students' cognitive understanding and knowledge of the subject. Indeed, van Gog, Paas, and van Merriënboer (2004) argue that worked examples alone are not effective in supporting students' acquisition of meaningful knowledge if process-oriented information such as the rationale of the solution steps and heuristics are not provided. So a delicate balance between the extent of the solution steps to be given and how much for them to think and write about is one of our major considerations when we prepared the guided-exercise. For example, Figure 1 shows a problem in the first set of the guided-exercise, heuristic scaffoldings in the form of questions such as "can you find an easy way to sum the terms in the brackets?", "What is your next consideration?" Instructional explanations and cognitive scaffoldings such as "To use the theorem, you need to establish inequalities. How would you begin?" and conceptual scaffolding such

as "Thus $\frac{1}{(n+1)^3} \geq \frac{1}{(n+r)^3} \geq \frac{1}{(2n)^3}$ for $r = 1, 2, 3, \dots, n \Rightarrow \sum_{r=1}^n \frac{1}{(n+1)^3} \geq \sum_{r=1}^n \frac{1}{(n+r)^3} \geq \sum_{r=1}^n \frac{1}{(2n)^3}$ " is shown in Figure 1.

2. Find $\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \dots + \frac{1}{(2n)^3} \right)$.

Suggested solution

Can you find an easy way to sum the series in the brackets? _____.

If not, what is your next consideration? _____ (One theorem you have learnt)

To use the theorem, we need to establish inequalities. How would you begin? _____

Observe that $(n+1)^3 \leq (n+r)^3 \leq (n+n)^3$ if $r = 1, 2, 3, \dots, n$.

Thus $\frac{1}{(n+1)^3} \geq \frac{1}{(n+r)^3} \geq \frac{1}{(2n)^3}$ for $r = 1, 2, 3, \dots, n \Rightarrow \sum_{r=1}^n \frac{1}{(n+1)^3} \geq \sum_{r=1}^n \frac{1}{(n+r)^3} \geq \sum_{r=1}^n \frac{1}{(2n)^3}$

That is (please evaluate the summations) _____

[Note: summation is over r and NOT n . What does this mean? _____]

Since $\lim_{n \rightarrow \infty} \frac{n}{(2n)^3} = \lim_{n \rightarrow \infty} \frac{1}{8n^2} = 0$ and $\lim_{n \rightarrow \infty} \frac{n}{(n+1)^3} = \frac{1}{8}$ [Why? $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 3n + 3 + \frac{1}{n}} = \frac{1}{8}$]

By _____ Theorem, $\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \dots + \frac{1}{(2n)^3} \right) =$ _____

Can you find $\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} \right)$? _____.

If there are infinite number of terms in the brackets and each term tends to zero, can we conclude that the sum of all these terms is still zero? Why? _____

Figure 1. An incomplete worked-example with instructional scaffoldings.

We also encourage students to do "self-explaining" and reflection by asking questions to elicit their reasons and explanations of what they are doing. As far as possible, "self-explaining" feature was incorporated into the guided-exercise to help students to monitor their procedural and conceptual thinking. From past year teaching experience, we noticed that students had a tendency to compute

$\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \dots + \frac{1}{(2n)^3} \right)$ by taking limit term by term to conclude that the

answer is zero. So, we deliberately ask the students to find $\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} \right)$

which consists of only 2 terms and both terms tend to zero when n tends to infinity. We direct their cognitive focus on the concept of taking limits of infinite number of terms—to make them think and explain why it does not make sense to add an infinite number of very small terms. So, in our guided-exercise worksheets, we insert blanks for students to fill in so as to channel students' cognitive domain into a reflection and self-explaining mode.

(II) Sequencing of questions

In order to design the guided-exercise, we need to understand the learning difficulties faced by students and hence scaffold their learning by way of a gradual increase in difficulty in sequencing of the questions. In addition, the choice of the partially worked-out examples in our guided-exercises should address the difficulties and misconceptions students may encounter and increase their mathematical self-efficacy in doing the weekly tutorial problems, independently without any form of guidance. In Calculus, students require both the mathematical procedural ability and conceptual understanding and knowledge to solve problems successfully. In Figure 2, we deliberately arranged the task of finding $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x+1} \right)^x$ to be the first problem for students.

1. It has been proved in the lecture that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$ or $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$, find $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x+1} \right)^x$.

Possible solution: Observe that $1 + \frac{1}{x}$ is actually $\frac{x+1}{x}$.

How would you make $\frac{x+2}{x+1}$ similar to the above, a constant plus a fractional term?

_____.

$$\lim_{x \rightarrow \infty} \left(\frac{x+2}{x+1} \right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{(\quad)}{x+1} \right)^x$$

$$= \lim_{x \rightarrow \infty} \left\{ \left(1 + \frac{1}{x+1} \right)^{(\quad)} \left(1 + \frac{1}{x+1} \right)^{(\quad)} \right\}$$

=

= e [Please show the necessary steps to arrive at this answer]

Is the indicial term same as the denominator? _____

Can you apply the result $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$? Why?

_____.

What must you do?

Let $y = x + 1$ so as $x \rightarrow \infty$, $y \rightarrow (\quad)$.

What do you observe now? _____

Figure 2. Solution steps with prompts in Guided Exercise 3.

This was to emphasise the importance of applying and highlighting the “beauty” of the result $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$ which the lecturer would have had explained and proved during the lecture. Instructional support such as “How would you make $\frac{x+2}{x+1}$ similar to

the above?” was given so that students could apply the procedural knowledge of converting the fractional term into one that the given result could be applied. The sequencing of the solution steps and the questioning prompts for student self-explaining (e.g., “What do you observe now? ”) have to be carefully crafted with reference to the anticipated students’ learning difficulties.

(III) Adaptive variability

According to a study done by Pass and van Merriënboer (1994) on secondary school students learning Geometry, greater variations in the worked-examples given within a lesson supplemented with instructional guides benefited students more than those who worked on less-varied worked examples in terms of knowledge transferring. This is in agreement with Dienes’ (1960) principle of variability in the learning of mathematics, which recommends varying the mathematical (also perceptual) details of the example while retaining some structural characteristics so that the learners may notice the essential similarity among the illustrative examples. This, to us is a form of scaffolding. As such, when we designed the guided-exercise, we deliberately created problems which looked slightly different. For example, in Figure 3, the region R in Question 1 is defined by a set of points based on the set builder notation whereas in Question 4, the region of the double integration is defined by the integration limits.

Not only did we craft different types of questions and sequence the problems according to the difficulty level as shown in Figure 3, we also encourage variability in solutions. For example, in the solutions to this problem : “Test the series $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ for convergence” in the topic of Infinite Series, we start off by prompting the students to check the limit of the sequence $\left\{ \frac{1+\sin n}{n^2} \right\}$ to ensure the students understand why they need to proceed with other tests. We produce two partially worked-out solutions, one using the Limit Comparison Test with $a_n = \frac{1+\sin n}{n^2}$ and $b_n = \frac{1}{n^2}$. Students are reminded to explain why it does not work. Next, we put in a heuristic scaffolding to guide the students choosing a suitable value of p in $b_n = \frac{1}{n^p}$ so that the Limit Comparison Test shows the series is convergent. In addition, we want the students to think of any alternative solution or shorter solution before we suggest using the inequality $-1 \leq \sin n \leq 1$ to arrive at the answer based on the Squeeze Theorem and the Comparison Test with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This problem in fact engaged some students in thinking why the use of $b_n = \frac{1}{n^2}$ in the Limit Comparison Test and Comparison Test produces different outcomes.

1. Evaluate the integral $\iint_R x \cos xy \, dA$ where $R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \pi/2, 0 \leq y \leq 1\}$

Solution
 First of all, you need to identify the region represented by the set R . It is a _____.

So the given double integral $\iint_R x \cos xy \, dA$ is actually the same as $\int_{y=0}^1 \int_{x=0}^{\pi/2} x \cos xy \, dx \, dy$.

[Note: write down the correct limits of integration - the inner integral is with respect to x]

That is $\iint_R x \cos xy \, dA =$ _____

Take a look closely, $\iint_R x \cos xy \, dA = \int_{y=0}^1 \int_{x=0}^{\pi/2} x \cos xy \, dx \, dy$ requires integration by parts.

So we prefer $\iint_R x \cos xy \, dA =$ _____

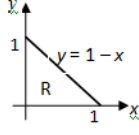
4. Use the transformations $u = x + y$ and $v = y - 2x$ to evaluate the integral $\int_0^1 \int_0^{1-x} (y - 2x)^2 \sqrt{x + y} \, dy \, dx$.

Solution: As it is, can you find the inner integral easily? So what must you do? _____

Yes, find out what the region of integration is. Integration limits for y is from 0 to _____ and limits for x is from 0 to _____.

Draw the region R which is a _____. (Pause here, draw the region first before you continue)

What must you do if you want to transform the region R onto a u - v plane? _____



Let us solve x and y in terms of u and v (using $u = x + y$ and $v = y - 2x$)

$x = \frac{1}{3}(u - v)$ and $y =$ _____

x - y equation for region R	u - v equation for new region	Simplified u - v equation
$x + y = 1$	_____	$u = 1$
$y = 0$	$\frac{1}{3}(2u + v) = 0$	_____
$x = 0$	_____	$u = v$

Next, draw the transformed region on the u - v plane.

Figure 3. Different variations of multiple integrals.

Results

As it was our first attempt at using incomplete worked-example with instructional scaffolding, we surveyed our students to find out their reactions to the guided-exercises. The end-of-course instructional survey was administered to the student teachers during one of the revision lectures two weeks before the semestral examination. The survey questionnaire captured the student teachers' demographic data and their perception of the various aspects of the Calculus II programme, one of which is the guided-exercise.

The participants of this survey were the 127 Year Two student teachers (84 female and 43 male students) of which 36 were enrolled on the Bachelor of Science (Primary Education), 79 on the Bachelor of Science (Secondary Education) and 11 on the Bachelor of Arts (Education). In the survey questionnaire, one of the areas we were looking at was whether this new initiative in our instructional programme was useful for students. The survey consists of 4-point Likert type questions (1 = Strongly Disagree and 4 = Strongly Agree).

Table 1 shows the responses from the 101 participants, which is 80% of the cohort. With regards to whether the guided exercises were useful for their learning, about 90%

of the students responded with Agree or Strongly Agree as indicated by items Q3A1 and Q3A2. In addition, it is not surprising to see that students who did better in the end-of-exam tended to respond favourably to the guided exercises as we found that there were significant positive correlation between the Calculus II results and the responses to Item Q3A1, Q3A2 and Q3A3 (p values are .029, .043 and .002 respectively). One undesirable observation was that the mean score for Item Q3A3 was quite low (mean 2.44) – due to only about 40% of the student teachers did the guided-exercises before their tutorial sessions. We believe that students’ lacking self-discipline and poor time management may be the reasons contributing to this disappointing percentage despite the benefits perceived by the students. The implication is that our students need to be closely monitored and tutors may have to make the guided-exercise a compulsory task for our future batch of students.

Table 1: Means and Standard Deviations of Students’ Survey Response

Item		1(SD)	2 (D)	3 (A)	4 (SA)	Mean (SD)
Q3A1	The guided exercise helps me to understand the topic better	1.0%	10.0%	52.5%	36.5%	3.25 (0.669)
Q3A2	The guided exercise helps me to reinforce the concepts	1.0%	7.9%	53.5%	37.6%	3.28 (0.650)
Q3A3	I usually do my guided exercises before the tutorials	14.9%	46.5%	24.7%	13.9%	2.44 (0.899)

The survey instrument included a free response section for students to comment on the guided exercise. Examples of their feedback on our guided-exercise are shown below.

- Useful to build foundation
- Useful for understanding concepts
- Very good
- Guided exercises gave solutions
- good guidance
- Guided exercises emphasized the important concepts required for the topics
- I practiced most of the questions in the guided exercises. It helped me to understand the concepts better

In particular, Student A commented that “guided exercises were good and the step-by-step promptings help me to go through the thinking process even though there were questions I didn’t know how to do It boosts confidence as the questions were more manageable compared to tutorials”. Another student B said that “guided exercises were indeed useful in terms of helping me to consolidate my revisions”. It is worth noting that Student A and B were among the weakest (in terms of Calculus I grade), these students showed great improvement in Calculus II result – an improvement of 4 grade points. That they were most positive about the guided exercise may indicate the usefulness of the exercise for them. It appears that the guided-exercises have benefitted at least two weaker students.

Conclusions

This paper describes our rationale, design and use of the guided-exercises as an instructional strategy to scaffold and enrich student learning. We believe that our guided-exercise worksheets in which incomplete worked-out examples were specially crafted with ‘interactive blanks or prompts’ would reduce students’ cognitive load and

misconception. It is our objective that after working through the guided-exercises, students would gain confidence and improve their conceptual and procedural understanding and mathematics self-efficacy which would better prepare them attempting the tutorial questions which are usually non-structured and non-guided. However, one limitation of our exploratory study is that we were not able to create controlled and experimental groups or conduct a pre and post test to investigate how effective this particular instructional initiative is. Nevertheless, the data from our preliminary study do give us an indication that the guided-exercise could supplement and enhance students' learning in the Calculus course. This study has given us enough positive students' feedback to encourage us adopting the guided-exercise for our next batch of students. It certainly gives us an impetus to carry out future research on how the guided-exercise approach can be further improved to enhance students' learning.

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The Transition From High School to University: The University of Queensland Perspective

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The characteristics of students in Australian undergraduate university mathematics courses have changed markedly in recent years. This change can be attributed to a number of factors. First, many universities have altered their entry requirements in a bid to attract students, by dropping prerequisites for enrolment and allowing students to study equivalent subjects once they enter university. Second, university students have more diverse backgrounds, both cultural and academic, than ever before (see, for example, [1-8]). As such, there is considerable interest in the transition from secondary to tertiary mathematics, particularly the first year of students' tertiary study. This paper gives an overview of the research into the transition from secondary to tertiary mathematics, and details the strategies The University of Queensland has used to assist students in this transition.

Keywords: secondary–tertiary transition in mathematics; bridging (remedial) courses; first-year mathematics; diagnostic testing

Introduction

The characteristics of students in Australian undergraduate university mathematics courses have changed markedly in recent years. A decrease in enrolments in advanced mathematics subjects at high school has meant that students are now entering university with weaker mathematics backgrounds. A move from elite to mass education, both in Australia and overseas, has led to students having more diverse backgrounds, both cultural and academic, than ever before. As such, mathematics and engineering departments have had to change the structure of their courses and programmes to incorporate these students. This paper begins by reviewing the recent research into the transition from secondary to tertiary mathematics, explores the reasons behind the decrease in high school advanced mathematics enrolments, and details the strategies The University of Queensland has used with its first-year students.

Literary review

The transition from secondary to tertiary mathematics education has been studied in considerable depth in the past two decades. There have been several ways to approach the study of this transition, ranging from the basic to quite complex.

Students' mathematical difficulties at university are a common focus for research (see, for example, [7-9]). Difficulties can be easily identified either through diagnostic testing, observation of class work or analysis of exams. This then often leads to the proposition of new teaching or intervention strategies. Some lecturers may choose to completely re-teach weak areas from the beginning, while others may teach the work again, but by linking it to students' current knowledge [7].

A more complex view is to look at the transition in terms of thought processes, proof and mathematical communication. In [10] mathematical thinking is classified in

two ways: practical and theoretical. Practical thinking is based on the logic of action whereas theoretical thinking is characterised by organised system of concepts. [11] divides mathematical reasoning into two categories: plausible and based on students' own experiences. High school and first-year university students typically fall into the practical thinking category, using familiar strategies to solve a limited range of problems. The move to a more abstract way of thinking and being able to adapt a known approach to solve an unfamiliar problem can take a lengthy period of time. [12] describes this shift from the surface learning to deep understanding as a move from elementary to advanced mathematical thinking. High school students often do quite well in mathematics by remembering and performing algorithms, without any deep understanding of the mathematics. Once they enter university these students can have considerable difficulty with questions that are only slightly different to ones they have experienced at school. [13] categorises this area of transition research as the epistemological and cognitive perspective.

While proof is taught in many high school mathematics courses, studies have shown that only a small number of students are able to build consistent proofs by the end of high school ([14]). Students may encounter proof in high school mathematics, but there is a clear difference between this and proof in tertiary mathematics.

[13] also categorises the transition research into two other perspectives: socio-cultural, and didactical. Not only do students meet more abstract concepts and more formal ways of thinking and reasoning, but the teaching methods and contexts are different at university compared with school. For example, at high school students often work in small groups, and teachers know the students well. In the university environment students are much more on their own, sitting in large lecture theatres of up to 400 students, and in tutorials up to 30 students with one tutor.

[14, 15] takes a similar approach, finding three general approaches to the research into the transition: thinking modes and knowledge's organisation; proof and mathematical communication; and didactical transposition and didactical contract. The view of [14] is that the transition is not something that happens between the end of high school and the beginning of university, or the end of high school through to the end of the first year of university, but rather a period of four years: two years before entering university and two years after. As such, there is a considerable time period over which research can be undertaken.

The Australian situation

The number of students studying higher level mathematics in Australian high schools declined in the mid-late 1990s and early part of this century. The proportion of Australian students taking advanced mathematics dropped from 14.1% of the Year 12 student population in 1995 to 11.7% in 2004 ([16]). The decrease was more marked in Queensland, the state of interest in this paper. The start of this decline coincided with universities removing prerequisites from their engineering courses, yet there are other reasons as well. The *Maths? Why Not?* Report ([17]) summarises mathematics teachers' and school counsellors' beliefs as to why students were not choosing to study higher level mathematics in their last two years of high school. The reasons stated can be grouped into five categories:

1. School influences
2. University influences
3. Sources of advice influences
4. Individual influences
5. Other influences

School influences can include the decrease in mathematically qualified teachers and consequent lack of depth in teaching approaches (especially in the junior secondary years), the increase in subjects offered that are in competition with mathematics, timetable restrictions, and composite classes. University influences include the removal of prerequisites, changes in degree structure, and more flexible and fewer compulsory courses. Source of advice influences can include guidance counsellors, teachers, friends in the same year level, and job guides. Individual influences include perceptions of ability, interest, and students' experience of junior secondary mathematics. Finally, other influences could be gender, parental aspirations, parental and public perceptions of the nature and value of mathematics, and understanding of career paths.

The *Maths? Why Not?* Report states that students' perception of their own abilities, their interest and liking for higher level mathematics, their perception of the difficulty of higher level mathematics, their previous experience and achievement in mathematics, and finally their perception of the usefulness of higher level mathematics were the most common responses from mathematics teachers and school counsellors. In addition, the greater appeal of less demanding subjects, parents' expectations and aspirations, composite classes, and non-mathematics trained teachers teaching mathematics were also significant reasons.

The University of Queensland

The University of Queensland (UQ) is Queensland's oldest university and is a member of the G08 and Universitas21. Traditionally it has attracted the brightest students in Queensland and northern New South Wales and as such has not had a strong focus on the transition from secondary to tertiary learning environments.

UQ had (and still has) higher entry score requirements than other universities for many programmes (due to the popularity of the University) and had strict prerequisites that had to be studied before enrolment. Since the move from elite to mass education, many prerequisites for courses have been removed, with students able to study these former prerequisites once they are at university. The University introduced bridging courses: these courses typically cover high school content of science and mathematics courses, yet they only run for one semester compared to two years at high school. As such, it is impossible to teach the same amount of content, and, importantly, students do not have as long a time period to understand and consolidate the material, and develop automaticity and fluency. Advanced mathematics was one high school subject that no longer had to be studied at school in order to enter programmes such as engineering. Once this prerequisite was removed the University introduced an advanced mathematics bridging course, predominantly for engineering and science students. This bridging course is taught by the mathematics department.

UQ's mathematics department is very much a service teaching department, teaching into the Bachelor of Engineering and Science programmes. The number of students studying engineering and science (non-mathematics-based degrees) far outweighs the number of students majoring in mathematics. As such, mathematics staff work closely with engineering and science staff to produce the best possible outcomes for students.

Until the mid-1990s the first-year engineering cohort at UQ was very different to the current cohort. At school, students had to have studied both intermediate *and* advanced mathematics, plus chemistry *and* physics, in order to enter engineering. Since the mid-1990s, students have only needed intermediate mathematics plus chemistry *or* physics to enrol. (UQ was the last Queensland university to drop the advanced mathematics prerequisite for engineering.) As such, only 60% of recent first-year engineering students have studied both intermediate and advanced mathematics at

school ([18]). This has left 40% of students entering engineering without two years of further integration, matrices, vectors, sequences, series, and complex numbers, all important topics for engineering.

The number of students choosing engineering in the 1990s was also very different. In 1991 there were 400 first-year students; in 2008-2011 there were 970, with the Bachelor of Engineering being the University's third largest program in terms of student numbers. This increase in numbers has meant an increased diversity of backgrounds, knowledge and abilities, but there has not been a significant change in the way that engineering and mathematics is taught. In order to better understand this new group of students, in 2007 staff decided to reintroduce diagnostic testing.

Diagnostic testing

Diagnostic testing had been conducted on first-year engineering cohorts at UQ from 1972-1994 ([19]). The test was the same every year with questions covering the Years 11 and 12 intermediate and advanced mathematics syllabi. The results showed quite a considerable decline in the matrices and vectors component over the years. The 2007 diagnostic test was quite different. Questions came from both the Senior Intermediate Mathematics Syllabus and the Queensland Years 1-10 Mathematics Syllabus (specifically topics which form the basis for the senior secondary topics). A typical course of study in Queensland Senior Intermediate Mathematics can be found in Table 1.

Table 1. Typical Queensland Intermediate Mathematics Outline

Year 11 Term 1	Fundamental concepts, applied statistical analysis, periodic functions and applications
Year 11 Term 2	Functions – limits, composite, inverse, non-linear
Year 11 Term 3	Periodic functions and applications, exponential and log functions
Year 11 Term 4	Rates of change, optimisation using derivatives, applied statistical analysis
Year 12 Term 1	Rates of change, optimisation using derivatives, introduction to integration – numerical
Year 12 Term 2	Periodic functions, exponential and log functions
Year 12 Term 3	Exponential and log functions, optimisation, introduction to integration – area under & between curves, rates of change – log functions, derivatives & graphing
Year 12 Term 4	Applied statistical analysis, optimisation

Test questions involved purely mathematical calculations as well as worded real-life problems. The test was given to all first semester advanced mathematics bridging students (n=457) and Calculus and Linear Algebra 1 students (n=583) in their first lecture. The advanced mathematics bridging students were students who had not studied advanced mathematics at school, yet needed the course for their programme. Calculus and Linear Algebra 1 students have usually studied intermediate and advanced mathematics at high school (or completed the advanced mathematics bridging course at UQ). Both first semester cohorts are typically made up of first-year engineering students (17-18 years old) who completed high school in Queensland.

The test was a pen-and-paper test with no prior notice. Students had approximately 20-25 minutes to complete the test and were asked not to use calculators. Students did not need to show working but had three options when answering each question. They could write their answer in the box, or tick one of two boxes: “never seen” or “can’t

remember”. One reason these two options were included was to discover if some students, particularly the non-Queensland students, had not seen some of the topics before. The second reason was to gauge which topics the students felt comfortable in answering and which they did not. An explanation of the test was given to students beforehand, which included students being told that if a question looks familiar but you cannot remember how to solve it, then tick the “can’t remember” box and move on. A summary of test items is given in Table 2.

Table 2. Summary of Test Items

Question	
1. Write as a single fraction $\frac{3}{x} + \frac{5}{x+2}$	9. Given the right-angled triangle below (picture supplied), state the value of $\cos \theta$.
2. Solve $5 + \frac{x}{2} = 2 + x$	10. A surveyor standing at a point B, 40m from the base of the tower, has measured the angle to the top of the tower as 60° (pictured supplied). Write an expression for the height of the tower in terms of the angle.
3. Expand and simplify $(2x - y)^2$	11. Let $f(x) = x^2 - \sqrt{x}$. Determine $f(4)$.
4. Factorise $9x^2 - 64$	12. When is $P(t) = t^2 - 6t + 16$ a maximum?
5. Solve $x^2 + 6x + 8 = 0$	13. Determine the first derivative of $f(x) = xe^x$
6. Simplify $(x^{1/2} \times y)^2 / x^2$	14. Determine the first derivative of $f(x) = \sin(7x)$
7. Evaluate $\log_3 9 + \log_4 2$	15. Evaluate the integral $\int \sqrt{x} dx$
8. You need to make 500mL of a solution that contains 10% (by volume) hydrochloric acid (HCl). What volumes of pure 100% HCl and distilled water do you need to make this solution?	16. Evaluate the definite integral $\int_0^2 (-2x + 3) dx$

Results

Students’ answers to Question 1 were perhaps the most surprising. Only 27% of the advanced mathematics bridging students and 57% of the Calculus and Linear Algebra 1 students answered correctly. The most common wrong answers were $8/(2x+2)$ (added numerators and added denominators, 8% of students) and $10/(10x+2)$ (added 2 to both the numerator and denominator of the first fraction, 6% of students). Four percent of students had the correct denominator but incorrect numerator (eight different numerators overall). Percentages of “can’t remember” responses were quite high for Questions 7 (logs), 12-16 (differentiation and integration). These were questions on topics that students had seen only in Years 11 and 12.

An in-depth analysis of the 2007 test results can be found in ([7,20]); however, students who had studied intermediate and advanced mathematics subjects in high school performed better on all questions than those who had just studied intermediate

mathematics, and students performed considerably better in topics to which they had more exposure. Questions on calculus, an area only studied in Years 11 and 12, had the lowest success rate. However, the results suggest that for *both* groups, students' understanding of the topics most recently studied, in this case, differentiation and integration, appear not to have been strongly consolidated, with students not having developed automaticity and fluency. In addition, the results suggest that students also have difficulty with topics they first experienced in primary school (e.g., fractions). There appeared to be a relationship between students' mathematics performance at high school and their performance in the quiz. This is consistent with the research in [21, 22] which showed that performance at the tertiary level is dependent on high school performance. While there was a high range of scores and the standard deviation was slightly higher than the other categories, Queensland students who received a Very High Achievement in intermediate mathematics at high school performed on average better than students who received a High or Sound Achievement.

Mathematics and engineering staff were not particularly surprised with which questions the students had difficulty (apart from fractions) but the low percentages of correct answers were surprising to many. For the advanced mathematics bridging course students, no question was done better than 80%, and the logs, differentiation and integration questions were all below 30%. The Calculus and Linear Algebra 1 students had a top mark of 88%, but the logs, differentiation and integration questions were between 22 and 50%. Even the Calculus and Linear Algebra 1 students who obtained a Very High Achievement in advanced mathematics at high school only achieved on average 50% success in the quiz.

For those questions with which students had considerable difficulty, it was interesting to note the range of responses. The high percentages of "can't remember" responses indicate that students have seen the questions before; however, either cannot remember how to do them or do not feel confident in attempting them. The latter may be connected to the mathematics anxiety research with students perhaps not attempting the question for fear of failure ([23]). Offering a "can't remember" option also allows students to tell lecturers that this is not unseen material; they may have "known" it once, but their knowledge is fragile and needs further strengthening. This may provide opportunities for teachers to build or strengthen students' understanding rather than teach the work from the very beginning. This provides richer and more sophisticated interpretation on students' understanding than simply asking them to do the calculation.

What did we do?

Given the time taken to mark the 1000-plus tests, little feedback was given to the students, other than the test and solutions being posted on course websites. Online tests were considered for 2008; however, it was not possible to design one in time for the beginning of Semester 1.

In late 2008 a team of engineering and science academics received a grant to design an electronic diagnostic test to assess knowledge of high school level maths, physics, and chemistry, and also the ability of the students to apply this knowledge. Survey Monkey was used to design and run the test; however, limitations on entering mathematics symbols meant that the test was multiple-choice, with carefully chosen distracters. Also included were the "never seen" or "can't remember" options.

The test was run just before the commencement of Semester 1, 2009, again with no prior notice. This time students could access the test at university or at home. The participation rate was disappointing (n=388), just over a third of the cohort. Most of the questions were the same as the 2007 test. Given the poor results in the algebraic

fractions and calculus questions in 2007, it was decided that four new questions would be added to the 2009 test:

1. a numerical fractions question – Write $\frac{2}{3} + \frac{3}{4}$ as a single fraction
2. a simple differentiation question – Determine the first derivative of
3. $f(x) = x^3 + 2x^2 - 7x + 4$
4. a simple integration question – Find the integral $\int (3x^2 + 4x - 1)dx$
5. find the area of the shaded region that is bounded by the curve, the x -axis, the y -axis and the line $x=3$, giving the answer to one decimal place (picture provided)

2009 results

While the raw percentages for most questions were better than the 2007 test, the order of difficulty remained the same. That is, questions on topics that students had first seen in Years 11 and 12 were not done as well as topics to which students had had longer exposure. Unfortunately it was not possible to analyse students' responses in terms of how many mathematics subjects they studied at high school. A re-test ($n=103$) was undertaken at the beginning of Semester 2, 2009, to see how students performed in comparison to the beginning of the year. Improvement was seen in all questions, particularly in solving a quadratic, chain and product rules, composition of functions, and definite integral.

What did we do?

Feedback to students on the test was limited to the correct answer appearing after each question was answered. Staff teaching the advanced mathematics bridging course did go through several of the test questions in the first lecture of semester, impressing upon students the importance of being able to integrate. Online and paper resources were developed for topics in the advanced mathematics bridging course ([24]).

Midway through 2009 mathematics staff met with the engineering staff who taught dynamics. The dynamics lecturers had previously mentioned that students' integration skills were poor, but integration was taught at the end of semester in the advanced mathematics bridging course. A decision was made to reorder the topics in the mathematics course from 2010, bringing differentiation and integration to the start of semester and moving sequences and series to the end. Integration questions, particularly integration by substitution, on the end of semester examinations in the advanced mathematics bridging course were done poorly, so the hope was that this reordering of topics would not only benefit students in their dynamics study but also in their mathematics study.

2010

In addition to the reordering of topics, a new drop-in centre was opened in 2010. Running for two hours a week from the first week of semester until the end of the first week of exams, the aim of the drop-in centre is to help students with their conceptual understanding of mathematics. Run by a lecturer (the author), the centre is for students from three courses: intermediate bridging mathematics, advanced bridging mathematics and Calculus and Linear Algebra 1. This centre is in addition to the First Year Learning Centre which is staffed by postgraduate students and open for two hours a day Monday to Friday. Questionnaires and conversations with students indicate that a clear majority of students visit the First Year Learning Centre for specific help on assignment questions. The First Year Learning Centre can get very busy with two tutors helping up to 30 students from possibly seven different mathematics courses. I saw a need for a

different type of centre, one where students can receive help from someone who knows exactly where they are up to in the course, what they have studied previously and how it links to other areas of mathematics.

As someone who has worked on both sides of the transition (11 years as a high school mathematics teacher and six years lecturing first-year students), I have detailed knowledge of what difficulties students bring with them to university and as such, when a student comes to the drop-in centre, I am able to link the student's current problem to their previous study, thereby allowing the student to build on their understanding. The number of students visiting the drop-in centre ranges from one to 10 per day. This small number increases the opportunity for deep learning to occur. Informal feedback via comments on teaching evaluations and conversations with students indicate that the drop-in centre is assisting students with their mathematical understanding and development. In addition to the drop-in centre, a team of first-year lecturers received a grant to develop a study guide to complement the lecture notes for the advanced bridging mathematics course ([25]). Developed over a six-month period, it covers the main points from each of the nine chapters, explains them in detail and gives worked examples and practice questions.

2010 results

The same test as 2009 was run again ($n=475$, approximately one-half of the cohort) just before the commencement of first semester 2010. Feedback to students was greatly improved, with each student receiving a personalised report detailing their mark for each question (correct or incorrect), which first-year course(s) each question was important for, and for each question at least one website to visit in order to improve their knowledge and understanding.

The 2010 cohort was made up of students with higher overall high school results compared with the 2009 cohort, and they performed better in all questions. It was possible to analyse student performance according to how many mathematics subjects they studied at high school, with the students who had studied both intermediate and advanced mathematics performing considerably better than those who had only studied intermediate mathematics. As in 2007 and 2009, students performed considerably better in topics to which they had more exposure. A re-test ($n=107$) was undertaken at the beginning of Semester 2, 2010; however, only 55 of those 107 did the original test so comparisons are difficult to make. The retest did provide some insight though.

One question on the re-test was "Did the individual report that you received after doing this quiz in Semester 1 help you understand what knowledge was necessary for 1st semester?" Of the 55 students who did the original test, 29 responded 'yes' and 15 'no'. Several students who responded 'yes' said that the test showed what topics they needed to revise.

The dynamics teaching staff reported at the end of Semester 1, 2010, that their students seemed to be better prepared as a result of integration being taught earlier in the advanced mathematics bridging course. However, there appeared to be little change in the advanced mathematics bridging course students' understanding of integration as students still had considerable difficulty answering integration by substitution questions on the final exam.

2011

In 2011 a different computer system was used to run the competency test. This new system allowed staff to format questions and send personalised reports much more

easily than the previous one. The test questions were the same as 2009 and 2010, and multiple-choice was again used. Participation was much improved: 743 students (approximately 75% of the cohort). The results again were very similar: the same order of difficulty, with students who had studied both intermediate and advanced mathematics at school performing considerably better than those who had only studied intermediate mathematics. The drop-in centre continued to operate, with more students utilising it. Staff have continued to add more questions to their open-access online question and solution generator and are working closely with the engineering faculty to design the new engineering programme which begins in 2012.

Where to from here?

The results of the diagnostic tests from 2007 to 2011 have been quite consistent: students who had studied both intermediate and advanced mathematics at school performed considerably better than those who had only studied intermediate mathematics, and students performed considerably better in topics to which they had more exposure. However, as it stands diagnostic test gives teaching staff some idea of what the students can do, but it does not give staff much of an idea of what students *understand*.

In 2012 the plan is to run the test electronically, yet with as few multiple-choice questions as possible. Some of the questions will remain, others will be replaced with questions which allow students to demonstrate their understanding, as opposed to what they remember how to do algorithmically. One likely question, and one that was asked on a recent advanced mathematics bridging course exam, is to give students a graph with a local maximum and minimum (e.g., $y = x^3 + 3x^2 - 9x + 4$) and ask them on which interval(s) the derivative is negative. A very young student can be taught the algorithm to differentiate a function, but knowing where a derivative is negative is a higher-order question. Other possible questions include:

- giving students a graph of a function and asking which of four other graphs is the graph of the derivative (or integral);
- find the integral $\int_{-3}^3 |x + 2| dx$ ([26]);
- find the maximum slope of $y = -x^3 + 3x^2 + 9x + 27$ over $[-4, 4]$ ([26]).

Several questions require thinking about:

1. Should the diagnostic test be run in Week 2 or 3, once students' minds have been working for a few weeks? Should the test be run before semester and then again in Week 2 or 3?
2. Given that the test results have been consistent over a period of time, do we need to do the test? Are the students benefiting from it?
3. How can we improve students' integration skills? We have tried various approaches but final exam results are still poor.

Conclusion

The characteristics of students in Australian undergraduate university mathematics courses have changed markedly in recent years. Students are now entering university with weaker mathematical backgrounds and changes have had to be made to mathematics and engineering programmes to accommodate these students.

Diagnostic testing at UQ has confirmed recent research that the transition from secondary to tertiary mathematics is a complex issue. Students remember and understand little of their senior secondary mathematics in comparison to their junior secondary content, and this has serious implications for university staff. The

introduction of the drop-in centre, run by a lecturer as opposed to students or tutors, has been a success in assisting students' mathematical understanding, and has added to the literature on the transition. One question that everyone involved in mathematics must ponder is: How can we encourage more high school students to choose advanced mathematics?

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Math Circles: Innovative Communities for Doing Mathematics

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Math circles have a tradition going back over more than a hundred years in Bulgaria and Russia. In recent years, math circles have sprung up all over the United States. They are informal institutions outside the traditional systems of primary, secondary and postsecondary education in which students do, learn, and discover mathematics, generally under the guidance of professional mathematicians. The participants range from kindergarten age to mathematics teachers. This article describes some characteristic features of math circles and discusses the experiences of a specific math circle designed to connect high school age students with research mathematics and research mathematicians. This math circle serves as a new means to widen and strengthen the pipeline into higher mathematics. It is also a promising forum for developing novel curricula, testing new instructional methods, and for conducting teaching experiments.

Keywords: math circle, outreach, extracurricular

Introduction

Math circles have a tradition going back more than a hundred years in Bulgaria and Russia where, for a long time, it has been just as normal for school-age students to participate after school in a group doing math for fun in the evening, as in other places it is common practice to take piano or ballet lessons, or join the swim or futbol club. In particular in the United States, until very recently such math clubs have been a very foreign notion – the closest maybe being a chess club. But over the past twenty years there has been at first a slow, and, more recently, an explosive growth of math circles of many different kinds that are adaptations of the Bulgarian and Russian models. For detailed descriptions we refer the interested reader to the book [18] by Vanderwelde, the webpages [14] of the National Association of Mathematics Circles (NAMC), and, for a quick introduction, the wikipedia page on math circles [19]. Some math circles are small family-run enterprises that originated when immigrant parents decided to supply their children with mathematics activities that went beyond what was addressed in school, and then opened up the sessions to classmates of their children. Some math circles focus entirely on practicing for high level competitions such as the International Mathematical Olympiad [8], some are organized by teachers and targeted at teachers, and others are entirely run by students. A common feature is that they generally connect motivated students with professional mathematicians to do mathematics that is complementary to school curricula – i.e., it is fundamentally different from remedial work or tutoring to help with school work. Some math circles are entirely supported by volunteers, others charge substantial fees, or are supported by donations from individuals, companies, or educational institutions and government entities. Indeed, many math circles are loosely connected to universities and schools with math circles being considered informal outreach activities rather than formal classes. While there is a demonstrated need for activities provided by the many forms of math circles, it is a nontrivial task to launch and sustain math circles. The recent establishment of the National Association of Mathematics Circles (NAMC) [14] which is closely connected

with the Mathematical Sciences Research Institute (MSRI) at Berkeley as a national umbrella organization has much facilitated the sharing of resources and organizational experiences.

This article describes some of the background, and the rationale for various choices made when establishing the first math circle in one of the largest cities in the United States. A large part of the article illustrates mathematical topics and themes that were addressed in the last year by our math circle. The article discusses what went well, what needed special attention – but foremost the presented items highlight how it is feasible to connect motivated high-school students to advanced mathematics, including open problems, under the guidance of professional mathematicians.

Designing and Launching the Math Circle at ASU Tempe

Whereas on both the Atlantic and Pacific coasts of the United States math circles have been flourishing for many years, wide expanses in the inland areas remain much underserved. The Phoenix metropolitan area with a population of over 4 million, and growing at an extremely fast rate (doubling every 20 years) is an extreme example. With only one single major university serving the area, and a mix of struggling and outstanding new schools, this region lacks the established networks that cities like Boston or San Francisco have to provide students at all ages with the extracurricular challenges they desire. Over the years, many individual parents and teachers have approached the university, asking for help to supply students with challenging mathematical programs beyond what the school system can supply. But starting new outreach programs from scratch is a major undertaking. This is where the National Association of Math Circles [14] lends invaluable support: It connected us with a community of experienced math circle organizers from around the country, it provides detailed instructions, e.g., in [18], and it offers financial support. (NAMC, in turn, is partially supported by grants of the National Science Foundation of the United States.) To jumpstart the process, our school teamed up with NAMC to host a local “Julia Robinson Mathematics Festival” (JRMF) [10] in conjunction with the first “Circle on the Road” Conference [14]. Also meant to gauge the demand and support in the community, the JRMF brought well over 400 participants to the School of Mathematical and Statistical Sciences at the university who spent a full Saturday doing mathematics. Sessions were led by the most experienced math circle leaders from around the country and their apprentices, who were in town for the subsequent conference. Rather than teachers bringing students by the busload, the organizers considered it essential to change the culture and have families, parents bring students one-by-one and in small groups. This is a radical departure from the common model (we are in the Wild West, not Moscow, New York, or Sydney) where parents take their kids to the ball game, ballet class and swim club, but hand off the responsibility to the schools and teachers to take the students to museums or the science center.

The JRMF demonstrated a tremendous demand for exactly the kind of outreach activities that math circles provide. Indeed, desires for math circles with different target audiences and in different locales were voiced. One large company offered financial support for establishing math circles in small mining communities across the state. But with the limited resources (personnel) we have, there was no hope to meet all these expectations. Instead, we decided to establish one model math circle which shall spawn more across the region. As the only major university in this large metropolitan area (contrast this with e.g. the Boston area with more than 50 well-known colleges and universities), our school is the only one with a significant group of research mathematicians. (Community colleges, industry and government have a few isolated

individuals.) Consequently, our math circle specifically aims at providing the desired mathematical challenges for the most advanced high school age students whose needs cannot be met by anyone else in the region.

This is not the place to discuss in detail the many organizational challenges the creation of the Math Circle at ASU Tempe [15]. Indeed, our experiences closely follow those described in the “*Circle in a Box*” [18] and in discussions at the *Circle on the Road* conferences. The Math Circle at ASU Tempe is supported by the School of Mathematical and Statistical Sciences at the University and a seed grant from NAMC. In each of the last two semesters the math circle has held about ten weekly 90-minute meetings in the early evenings. Advertising, recruitment, and scheduling conflicts of busy high school students are common challenges.

Following the recommendations regarding social interactions made in [18], we reluctantly turned down numerous requests by parents to let their much younger children join. While occasionally, at some events younger students participated, and provided some of the most creative insights, it was immediate that the “*Circle in a Box*” was right, and social strains quickly appeared. We encouraged the youngest ones to come back in a year or two.

Unfortunately, it is difficult to achieve the desired diversity: The returning participants in our math circle predominantly come from select suburban high school districts which are less ethnically diverse than the metropolitan area. It is a constant challenge to broaden our reach to schools, teachers, and families who are traditionally less connected to the university. So far, we have been fortunate with a healthy gender balance - but this also is a result of constant conscious efforts in recruiting, classroom techniques and personalized follow-up. Over the last year almost 40 students attended at least some sessions, and about a dozen regulars rarely missed any. The levels ranged from fresh(wo)men to seniors in high school, and several home-schooled students.

Mathematical Program of the Math Circle at ASU Tempe

Our math circle specifically targets students like the high school freshman referred to us by her teacher who said that the student was too far ahead and the school could not provide enough challenging activities for her. While assuming a certain level of general mathematical maturity of the participants, we make every effort to not assume a specific level of completed classwork. This means, we do not assume that students have a specific preparation and level of mastery in algebra, geometry, understanding of functions, or calculus. Indeed, our goal is to provide a program that in some sense is *orthogonal* to secondary school curricula. It is a major challenge to design a curriculum that is coherent, that benefits the students, and into which students, parents and teachers *buy in*. Indeed, this raises immediate questions: If the topics chosen are important, why are they not part of the school curricula? If they are orthogonal to the school curricula, which are developed as a team effort by generations of experts, how can the topics form a coherent curriculum?

In the end, the topics at our math circle were dictated in large part by the research expertise of the available faculty and their expressed preferences. Our group agreed on using *problem solving* as a natural start to get everyone involved right away. Simple problems and mathematical puzzles as in popular books [3, 11, 20, 21] serve as reliable ice-breakers. The website of the NAMC [14] is a valuable resource for tested lesson plans and topics for a wide range of ages and levels of mathematical maturity. Our faculty team and the student participants who became regulars agreed that our circle NOT focus on preparing for mathematical competitions such as the American Mathematics Competition [1] of the Mathematical Association of America (MAA), the

Putnam examinations, or the International Mathematical Olympiad (IMO) [8]. Instead, except for the warm-up problems as mentioned above, we introduce students to problems which lead to ever new problems and eventually to entire research programs. The example of Ramsey theory discussed below is a prototype for such activity. Other than enjoying mathematics, developing deeper understanding and knowledge, skills, and preparing for postsecondary work in mathematics, a tangible goal for participants in our math circle is the annual INTEL Science Talent Search [7] (formerly called the Westinghouse High School Competition). Every year it features outstanding research performed by high school students, who often are mentored by university faculty, and it awards substantial prizes to its winners. While much of the work is in the biological, medical and physical sciences, every year the event also features notable mathematical research.

Rather than emphasizing problems to be completed competition-style, and which are put away when completed, our emphasis is to foster a reflective thinking style that is typical of research mathematicians, asking lots of questions, and always exploring new lines of investigation. Indeed, the experiences in our still young math circle suggest that this might well be one of the most valuable experiences our students take home from our meetings: Experiencing the methodical and deliberate thinking style of professional mathematicians that is so opposite to the almost always rushed habits familiar from doing scores of exercises in traditional school classes and homework. It is a slow process, but our students visibly changed from first being shy, then blurting out fast responses, to eventually mimicking the approaches demonstrated by our session leaders.

It is a real challenge for most research mathematicians to communicate to the general public what they are working on. One may argue whether it is any simpler to develop meaningful sessions which convey cutting edge topics to talented high-school students. This requires talent and much practice, and it is much harder in some areas than in others. The general agreement is that topics from discrete mathematics and geometry are generally more easily accessible and suitable for this environment, than topics in analysis. In the end, our choices were dictated by the limited supply of outstanding research faculty who were ready to work with high-school age students. The themes for the sessions of our math circle may be loosely grouped as follows:

- Discrete mathematics and graph theory: Starting with mathematical puzzles (see the next section), we introduced computational tools for experimentation (SAGE), and explored a side path to Markov chains. Following advanced examples of the pigeonhole principle, other sessions introduced topics from extremal graph theory and Ramsey theory.
- A live performance of Jugglematics [15] served as an introduction to modular arithmetic and topics from abstract algebra and elementary number theory with applications to cryptology. Surprisingly, the mathematically most advanced session which introduced ideas behind elliptic curve cryptography session had the most enthusiastic response from the students.
- A session led by the visiting director of the UCLA Math Circle explored a novel way of reasoning from mechanics to deduce theorems in geometry. A sequence of related sessions in geometry and topology started with polyhedral, extended the Euler characteristic to higher dimensions, and led to the arguments behind the Poincare-Hopf theorem which places stringent restrictions on the nature of vector fields on flows on manifolds (the hairy-ball theorem and the persistence of the red spot on Jupiter).
- A hands-on session on geometry, led by a retired teacher and national leader in secondary education, explored soap films using various wireframes, and

arrangements of multiple connected soap bubbles. This experimental session was followed by an introduction to the theory behind soap films and bubbles [9] and the famed proof of the double bubble conjecture [5, 12] which is well-known for the contributions made by undergraduate students.

Sample Problems in Discrete Mathematics and Graph Theory

Many well-known mathematical puzzles that may be found online or in popular references [3, 11, 20, 21] are excellent to immediately capture the students' attention. The following is a typical example that in its many variations naturally leads into higher mathematics such as Hamming codes.

A teacher announces to her N students that she will place one post-it note on each student's forehead, displaying one number from 1 to N . Each student cannot see her number, but all the other students can. Any number may be omitted or repeated any number of times. Once the notes have been placed the students are not allowed to communicate in any way with each other. Each is asked to guess her number and quietly write it on a piece of paper (not seen by the others). To collectively win a prize, at least one of the students must guess correctly. The students are allowed to discuss a strategy before the post-it notes have been distributed.

On first encounter, most consider this a perplexing, impossible task. It is even more impressive how relatively simple mathematics provides an amazing solution. This is not a place to give away solutions to famous puzzles, but we provide the hint to enumerate the students, use modular arithmetic and the pigeonhole principle. A research mathematician is not needed to help the students discover a solution and to verify it, but she will be needed to generalize the problem and strategy, and to deliberately reveal parts of the road to related recent and active research. Our math circle developed ideas related to Hamming codes in several sessions in increasingly sophisticated contexts.

Ramsey theory is another wonderful example in which the basic problem can be explained in a minute to most anyone, yet it takes just a few steps to famous open problems and active research

What is the smallest number N such at every party with at least N people contains at least one group of three people any two of whom know each other, or at least one group of three people any two of whom do not know each other?

This simple problem is easily solved by motivated students like those in our math circle, once they are given some suggestions to effectively translate this into a graph coloring problem for the complete graph on N vertices (popular colors are red and blue). After this warm-up our math circle continued to develop increasingly more sophisticated arguments that resolve the cases when the two threes in the statement are replaced by three and four, or by four and four. The asked-for numbers (size of the party) are known as $R(3,3)$, $R(3,4)$ and $R(4,4)$, respectively. The sessions were led by a Ph.D. student who was completing his thesis on extremal graph theory, and who thus could share his unique perspective taking the simple $R(3,3)$ party problem to unsolved open questions. A popular quote [4] attributed to the legendary mathematician Paul Erdős further motivated our students.

Aliens invade the Earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a pre-emptive attack.

Computer Experiments in Discrete Mathematics

Our math circle has a core of regulars who occasionally miss meetings due to conflicts. It is a challenge to provide sequences of sessions that explore specific topics in depth, but which do not make it impossible for new students to join. The students said they wanted short sequences of topics, each session leader running one or two meetings. Each new faculty has to find the right level – but the students meet many different researchers.

A related challenge was to get a common platform for computational experiments. In the spring term, the attendance was sufficiently stable so that we invested precious time to introduce the students to mathematical computing. We chose the package SAGE because it is free, it is what professional mathematicians use, and it is well suited to code the kind of algorithms that appear in our math circle. While most students had practically no prior programming experience, especially in mathematics (other than graphing calculators), one was an adept programmer in C++. But taking most everyone by surprise, the youngest in our group, a hitherto very quiet freshman in high-school was an experienced programmer in PYTHON (which is closely related to SAGE). It was fairly easy to get this group started with SAGE, and it completely changed their attitude towards doing mathematics: Computer experiments, collecting data, formulating hypotheses, and even brute-force checking of all possible cases became popular options and gained credibility – because the session leaders were well-published researchers who credibly demonstrated that this is the way they work. Along the way, the students became acutely aware of computational complexity, and at what stage brute-force calculations become infeasible. Not only $R(5,5)$ (see the preceding section) was seen in a new light, but this also prepared the stage for the later sessions on modern cryptography using logarithms in finite fields.

The following examples illustrate the use of SAGE computational experiments, which led the students to make conjectures that were proved in subsequent sessions.

Dice. Is it possible to label the faces of three dice with the numbers 1 through 18 (no repetitions) so that whenever your opponent selects a die, then you can choose one of the other two dice so that you will win (roll the higher number) on average? How should the faces of the dice be labelled so that your probability of winning is maximized. What is that the highest possible probability?

Airplane. An airplane has N seats, labelled 1 through N , and N passengers holding tickets also labelled 1 through N stand in line in some random order. The passengers enter the plane one by one. The first passenger is absent-minded and takes a random seat. Every other passenger takes a seat according to the rule: If her seat has not yet been taken, then she sits down in her assigned seat. Else she takes a randomly chosen seat that is still open. What is the probability that the passenger who enters the plane last will sit in her assigned seat?

Penney's game [16]. Vanessa and Miguel play a game that involves repeatedly flipping a fair coin whose sides are labelled Heads (H) and Tails (T). First Miguel picks a sequence of three possible outcomes, say, TTH (standing for Tails-Tails-Heads). Then Vanessa picks a sequence, say, THH. The winner of the game is the one whose sequence appears first. Say, with above choices, if the coin lands on H H T H T H T H H T T then Miguel wins. What should Miguel do to maximize his chance of winning? What should Vanessa do?

The first problem is accessible to high school students, yet it is a serious stepping stone to open problems in partially ordered sets. Computer experiments suggest a surprisingly simple solution for the second problem, which is then easily proved by strong induction (strong induction being essential). The third problem was popularized in the late 1960s, with variations of it being analysed in a contemporary research publications [5].

Sample Problems in Sessions on Geometry and Topology

While not directly linking to ongoing research, one sequel of exciting sessions spanned the entire sequence from counting at the kindergarten level to graduate level algebraic and differential topology. The eventual goal was the Poincare-Hopf theorem, which strictly constrains the behaviours of vector fields and flows on compact manifolds. Its beauty stems from connecting the purely topological concept of the Euler characteristic on one side to the analytic notion of index of a singularity on the other side. In colloquial conversations, this theorem is often referred to when explaining the persistence of the red spot on Jupiter, and more succinctly as the *Hairy Ball Theorem*. The latter says that it is impossible to comb the hair on a furry ball, or that at every time there must be two places on Earth where there is no wind. The Poincare Hopf theorem is typically a topic in graduate courses – but it is a popular item that is frequently the subject of illustrating mathematics to the nonmathematicians. Indeed, at the Julia Robinson Mathematics Festivals [10] a highly popular task was to explore possible flows on spheres by drawing flowlines on helium balloons (according to rules that encode continuity). The objective in our math circle was to develop the concept of topological invariants and then connect it to tangible applications. To develop the concept of the Euler characteristic, we started with physical models of the Platonic solids (and, while building the models, argued why there are only five). The counting of edges and faces on icosahedra and dodecahedra was surprisingly tricky with numerous double counts and undercounts. This immediately motivated upset students to instead use elementary mathematics (shared edges, number of edges at each vertex, etc.). We recorded the counts of vertices, edges, and faces in tabular form and the first observations were made: The sum of the numbers of vertices and faces is always two larger than the number of edges. Using interactive computer models available on-line, the students verified that this conjecture extended to various truncated polyhedra. Following the spirit of the classic discussion by Lakatos [12], (compare also Eppstein [2]) the group tried to prove the conjecture: This repeatedly demanded making better definitions, stating more precise hypotheses. Models of piecewise planar tori and double tori sharpened the arguments to the point where the classic theorem for dimension two was established to the satisfaction of all students: *The alternating sum of the number of vertices, edges and faces equals two minus twice the number of holes in the surface.*

A homework assignment explored the generalization of the Euler characteristic to higher dimensional simplices and cubes. Impressive counting arguments established that

the alternating sum of the number of higher dimensional faces is zero in even dimensions and two in odd dimensions (simplices and hypercubes in n -dimensional space) – discovering these important topological invariants of higher dimensional spheres. On the other side, as expected, none of the students had any knowledge of vector fields as mathematical items. But with interpretations such as wind and water flows the students considered them tangible objects. Mimicking the aforementioned drawing on helium balloons featured at the JRMF festival, it did not take much time to get some feeling for the possible simple singularities: sources, sinks, and saddles, and how pairs of a saddle and a sink (or saddle and a source) may be deformed away, whereas this is not possible for sink-source-pairs. Drawing vector fields (flows) on polyhedra was easy, and immediately the students took to the obvious candidates: Put a source on every face, a saddle in the middle of each edge, and everything flows into a vertex. Of course, the math circle sessions did not have the means to establish complete formal arguments, and there was a lot of hand waving. But under the leadership of research mathematicians, there never was any doubt about the inherent soundness of the arguments (it was just technical details that could be filled in at an appropriate later time). This was a challenging teaching experiment, but according to student feedback months later, it provided memorable and lasting deep impressions. In particular, the idea of topological invariants impressed several students, and all were astounded by how quickly one can go from kindergarten level counting to very deep mathematics. Following this experience, the author repeated an abbreviated version of this lesson plan in the first-year seminar for mathematics majors at the university, which is designed to introduce new students to the notion that after algebra and calculus the courses change dramatically. As expected, the math circle participants were more enthusiastic and were more creative and innovative – but the teaching experiment in the math circle helped the author prepare a better session for the formal university course!

Summary and Outlook

(Except for Bulgaria and Russia), math circles are a relatively new vehicle to promote doing and enjoying mathematics in the general population, especially among school-age students. The umbrella, guidance and networking provided and made possible by the National Association of Math Circles (NAMC) much facilitates the establishment of new math circles, avoiding having to reinvent everything from scratch. Given the special locale of our Math Circle at ASU Tempe, we developed a model of sessions led by a diverse group of research mathematicians who are experts in introducing problems in a language that is understandable by high school students, and in uncovering exciting paths from simple problems to open research challenges.

Our effort is too young to provide hard data on outcomes and efficiency. However, the students keep coming back in growing numbers. The faculty enjoy the sessions, which force them to rethink how to introduce material, and how to lead sessions without the usual safety net provided by the prior knowledge that graduate students are guaranteed to have due to completed prerequisite coursework. Already in this short time the Math Circle at ASU Tempe has proven to be an indispensable means to connect with the community as this is becoming the first place where curious students now make contact with university faculty.

Our goals for the medium term are to spawn new math circles that target various different groups and that have different thematic focus, that students who participated in our program make it to the final stages of international events such as the INTEL Science Talent Search [7], and that they show superior success in their post-secondary work.

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Learning Mathematics Pacific Style

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Learning mathematics and succeeding can be a challenge for many students. For Pacific students, this challenge is intensified with further factors which can affect their success. In this study, Pacific students studying mathematics at a university in the South Pacific, outside of New Zealand, were investigated to shed some light on how Pacific students studying mathematics away from their homeland might be benefited to improve their chances of success. The results highlight the responsibilities and obligations a strong cultural upbringing has on Pacific students; the importance of bilingualism; the different learning modes favoured; the importance of developing supportive relationships with their lecturers; and the support required from parents and family.

Keywords: Pacific students; mathematics education; learning support; tertiary students

Introduction

Helping students succeed in university mathematics is a matter which all mathematics teachers and mathematics educators should be concerned about. Success, however, is dependent on many varied factors and can mean different things for different people. For Pacific students, success at university can often be hindered by instances of personal hardships [1]. These include family pressures, health and housing issues, lack of personal motivation combined with the less disciplined tertiary lifestyle, and also institutional barriers of unfamiliarity with the university environment and academic work.

This study is motivated by a concern for the low participation and achievement of Pacific students taking mathematics courses at New Zealand universities [2]. Many mathematics educators share this concern and have introduced programmes and facilities to address the imbalance. Although some success has been achieved, the imbalance remains. The aim of this study is to help address this gap. By working with a group of Pacific students studying undergraduate mathematics at a university in the South Pacific outside of New Zealand, the study investigated the research question: what factors impact on Pacific students' learning mathematics at tertiary level? By conducting the research on their home ground, the aim was to shed some light on how Pacific students, who study mathematics at universities away from their homeland, might be benefited.

Although this study is set in the context of undergraduate mathematics students, we believe our findings are transferable to other subjects. In this article, we restrict our scope to highlighting the factors which emerged affecting Pacific students learning mathematics at tertiary level when they are attending university on their 'own ground'. We do not present solutions; instead we investigate the culture and context that surround Pacific students to improve our knowledge for helping them learn mathematics.

The paper begins with the background to education in the Pacific and the influence

that colonisation has had. Next, the methodology is described of how the data was collected and analysed. The results and analysis follow with six main themes presented from the data and a discussion on how these findings might inform practitioners and educators involved in teaching Pacific students, in particular mathematics.

Background

Education in the Pacific has been strongly influenced from European colonisation. This is reflected by being heavily teacher-centred and exam-driven with the focus on transmission of knowledge through language, in particular English, while not recognising the students' first language [3]. Although approaches to study and conceptions of learning have many features in common across cultures, noticeable to Pacific cultures is the tendency to emphasise collaborative and participatory styles, whereas Western cultures place weight on individual and competitive ones.

Culture can be generally defined as those beliefs, artefacts and practices that history has shown to be effective for the maintenance of a society and its future generations [4]. All cultures differ and in the context of teaching these differences help us to understand that, as Andrews [4, p.5] states "wherever we are located, forces, possibly beyond our consciousness, act to shape what happens in the classrooms. They determine what is to be taught, to who it is taught, how it is to be taught and where it is to be taught". Andrews argues that how mathematics is taught and how students learn, are shaped by the cultural norms which operate. Defining these norms however is difficult as culture is multi-layered. The behaviours that are observed and the values and beliefs that are reflected informing the behaviours, or the unarticulated assumptions that influence these, are difficult to evaluate and therefore easily contested.

Some researchers [3, 5, 6] contend that the authoritarianism of Pacific culture has been pervasive in education. This influence is reinforced, they believe, by the preferred learning style of rote-learning that persists and the reluctance to adopt another style. The authors claim that although egalitarianism, competition and individualism exist, in order to survive within the school system students adapt and by the time they enter university they are a product of the schools' institutional culture. Phan and Deo [6, p.372] describe this culture as the "notion of respect for authority, belief in interdependence and collectivism ... unquestioning acceptance of 'truth' and the entrenched silence as a *status quo*". Differing from this, they believe would be seen as a sign of disrespect and disobedience, and reflect on a student's upbringing. Consequently, the result is students are reluctant to ask questions in class, reinforced by their culture that figures of authority must not be questioned.

A large scale study in New Zealand identified factors that influence students having successful completion of tertiary qualifications [2]. Factors distinct to Pacific students were the effects of the interface between students and the institution, and the institution and community, around motivation and attitude. The study revealed that a high proportion of students were from families where tertiary educational experience is rare as well as a lack of role models for good academic habits. Many Pacific students admitted they did not try as hard at university, having a failure mindset, because their cultural differences were not acknowledged.

According to some researchers, the pressure from family groups for Pacific students to attend church, be present at family activities, and look after sick family members is considerable [2, 3]. Further, they also report that family, community, religious and work obligations frequently take precedent to study obligations and could impact enough for them having to abandon their studies if necessary. As well, peer groups were found to have enough influence to hinder academic deadlines and requirement; finances could be

a barrier to completing tertiary education as they may have to work to provide money for their families; there was a lack of support services or knowledge of what they were or how to access them; and language issues were highlighted for those whose English was not fluent or lacked the confidence to join in discussions. Positive factors found to increase retention included the availability of Pacific staff in institutions promoting a Pacific presence with positive role models, together with appropriate pedagogy and readily available information [2].

Another long-term study within a New Zealand university investigated how teaching practices in non-lecture contexts help or hinder the success of indigenous Māori and Pacific students [7]. This study identified that learning skills needed more clarity and that students needed to connect their experiences to people who believed in them and their dreams, gave them hope, and understood them as Māori or Pacific until they became independent. The findings signalled that success is more than just a grade and that “while some may think of academic support as being distinct from ‘pastoral’ support, these findings suggest that there is no clear distinction” (p.14). The authors point out that early childhood experiences, culture and language are pivotal to being a successful student. The use of Pacific nation languages and metaphors in academic support aided understanding of course content, student retention and encouragement. In other words, students who conversed with their tutor in both English and their Pacific nation language were able to gain a better understanding of words or terms by referring to their culture, customs or way of life, increasing their success [7].

Phan and Deo [5, 6] believe that Pacific students are at the *crossroads* in their learning “transgressing between the traditional forms of learning to the more contemporary ‘Western’ approaches” [6, p.380]. They consider that having a dual approach caters for students’ growing ambitions and socio-economical mobility, while incorporating both traditional and Western forms of learning. This type of approach they argue is a more culturally inclusive pedagogy.

Although these studies provide valuable information around education for Pacific students in general, our study aims to identify factors which impact on how Pacific students learn mathematics at a university on their home ground.

Methodology

The three authors worked together with the mathematics department at the university in the South Pacific, where the students were enrolled, to explore the question: what factors impact on Pacific students’ learning mathematics at tertiary level? The rationale was to develop a working relationship with the mathematics department for on-going collaboration. Using a mixed-methods approach, data was collected using a survey and interviews, followed up with many discussions with the mathematics department when two of the researchers, who came from New Zealand, were visiting.

Participants and research site description

The participants in this study consisted of an homogeneous Pacific group of 145 students recruited from undergraduate mathematics courses at a university in the South Pacific outside of New Zealand. Twelve students were from the first year of a bachelor degree programme majoring in mathematics, 118 students from a pre-degree foundation mathematics programme to qualify for university, and fifteen from a preliminary year that provides preparation for the qualifying year. These numbers were typical of the total numbers enrolled. Although there were only twelve students officially studying at

university level by New Zealand standards, no differentiation of students occurs at this university and all students are regarded and treated in the same manner.

Participants were invited to participate in a survey and could then choose to be interviewed later in the week replacing a cancelled tutorial. At the time of the surveys, the students were near the end of the first semester. Twenty-three students self-selected to be interviewed in groups of up to four students: seven from the degree, and sixteen from the foundation programmes.

Data collection

Data collection of both surveys and interviews were carried out by three researchers from a New Zealand university who were not known to the students. Two of these researchers were New Zealand/Pacific and one was a fluent speaker of the Pacific nation language spoken in that university. This latter researcher was not involved in the study but offered her interviewing skills as a native Pacific speaker. The surveys were distributed to all students at the start of a scheduled lecture. Four lectures in total were targeted. Students were given twenty minutes to complete the surveys which were then collected by the researchers. The students were able to ask questions prior to starting, or at any time while completing their surveys. The surveys were designed to obtain background information on the students, and to obtain an indication of how much time students spend doing mathematics each week, the help they receive, their preferred language for study, and the way they study mathematics with respect to the environment and culture that they live in. The surveys were collated and data recorded in an excel spreadsheet.

Due to the exploratory nature of this research, the main focus was on the interviews to provide in-depth case studies. Participants were interviewed in small focus groups which varied from 30-60 minutes depending on how many students in the group. Interviews in groups of three or four were used to encourage discussion, but some interviewed in pairs and two separately due to availability. Students were given the choice to be interviewed in their first language and a few students selected this choice. The students chose pseudonyms for themselves.

The interviews were audio recorded and transcribed, and were organised around questions about a typical day for the students at university, role models, goals, where they preferred to study, whether they studied alone or with others and whom these might be, and what their perceptions were about studying in other countries such as New Zealand. We used an open semi-structured format with the interviews to allow us to probe students' responses more closely if needed.

Data analysis

An important aspect of this research is the collaboration between the researchers who are themselves Pacific and work in this Pacific university, and the researchers who work within a New Zealand university. Although the academic mathematics staff of the participating university were not formally interviewed, they took part in discussions and provided guidance.

The student interviews were analysed qualitatively [8]. The transcripts were read separately by the researchers to identify excerpts and themes that were relevant to the research question using open coding. These were then compared and collated into categories to describe the student responses. Through an iterative cycle using axial coding these categories were analysed further in finer detail to identify the main themes and provide a description of what emerged.

Results

Six main themes emerged from the interviews. The first one provides a general context for understanding the background of Pacific students and how this impacts on the university expectations. The following four are more specific to studying mathematics at university, and the final one records the perceptions that Pacific students have about studying overseas, in particular New Zealand. Each theme is described in turn.

Students have to juggle family and community obligations with university expectations

For most mathematics students, a typical day at university would begin at 8am and finish between 2pm and 5pm, although a number of students had evening classes finishing around 8pm. Most students live at home in town or surrounding villages and commute by bus. Those living faraway and from neighbouring islands stay with extended families as one girl explained: “My village is far away from town. I stay with my aunty. I go home at holidays and for the big semester break”.

Without exception the students described their parents as strict, influencing them to develop good work and study habits. All those interviewed felt it was important to be accountable to their parents and therefore were motivated to do well. One student clarified that “my home environment is really good because Pacific parents are strict. They always tell their kids to go do your class work and do your homework”. Another student expressed gratitude that he was “really engrained with that attitude” from having strict parents.

Many of the students preferred the atmosphere of working at home if it was quiet, with no distractions, as it was easier to focus. One student said:

I prefer home. It is just me and my schoolwork. I get distracted so I need a silent place to work and at home I can think clearly. I can think to myself. At university I can't say to other students to stop making noise. So I prefer home.

Friends at university were seen as a great distraction and hard to resist:

They tend not to disturb me at home when I'm studying. At university it's like, you have your friends and many distractions ... we never settle down. We just go talk and move around, no focus, but at home it's just you working in your bedroom, just focus on whatever you want to do.

However, not all students found home life conducive to study with many distractions cited such as the TV, radio, play-station, or karaoke and their preference was studying at university. Others felt they had too many chores to do such as babysitting younger siblings or nieces and nephews; cooking; and cleaning; which impinged on their study time. Additionally, evenings and weekends for most included commitments with church activities such as youth and choir practices, and attending services. Additionally, a number of students belonged to sports teams, in particular rugby for boys:

There's so many commitments that I have made, you know, and I hardly have time to study. I baby-sit my younger brother. I attend choir. I play rugby after school.

Despite their varied commitments and obligations, none of the students complained about doing these, rather, they talked about their struggles to accommodate home expectations with their university ones.

Students had strong role models for doing mathematics

Many students described their role model for doing mathematics was a close family member, such as a parent, an aunt or an older sibling. In many cases this role model was also good at mathematics and they wanted to emulate them in this subject. One student told us: “my mum had some sort of degree for stats. She’s very good at it. My father, he took pure math. ... and so for me, you know, I aim to do as good”. Another said:

I needed someone to look up at, I needed something to inspire me and help me. I thought of my parents. I am the oldest child out of five children. So I need to set a good example for my younger siblings. And maths is interesting. ... so I chose them [my parents] as my role models.

And, another described how his father, who had passed away leaving eleven children, was his role model. All the children were having a university education because the father knew the importance of it “to promote us so we can have a better education and better future” his son said.

A few students commented on some TV shows that were role models for mathematics, in particular a programme called ‘Numbers’. One student explained that “it’s very encouraging ‘cause it’s cool seeing the main character applying stats and maths into solving the equations of accidents and stuff” and another said

people have to solve, find out the criminals through numbers. So that’s like really inspiring when I watch it ‘cause I can see them going through a lot of formulas, from one formula to another. So that makes me think, if they can do it I can do it too.

Mathematics was considered the hardest subject at university by all, but the students were keen to pursue it knowing it was a gatekeeper to potential jobs. Most were aspiring to be engineers, dentists, chemists and accountants as these were considered good paying jobs, were valued in their community, and always needed in their country. One student wanting to be a dentist explained: “I think that involves lots of maths, so I have to be so good at maths to achieve that career”. Another said “I think I’ll do more maths ‘cause I’m thinking engineering ... it requires maths”. One student wanted to do environmental science and expressed that

next year I’m going to pick up geography, economics, definitely math, and a little bit of chemistry. I want to be a counsellor for environmental science to advise people on how to use their resources. I mean, we are running out of fuel. In the next 50 years, we’ll be out of fuel so that’s where the idea came up.

The students had a very real understanding of their country’s economic situation and a strong social conscience. They planned to provide for both their families and their country.

Students prefer to work alone as well as collectively doing mathematics

The students emphasised the value of tutorials for understanding the lectures and interacting with the tutor as it was considered disrespectful to ask questions in the lecture. One student explained: “I make sure I attend tutorials ‘cause that’s where you’ll ask your questions, and that’s where they’ll practically cover everything in lectures”. The students reported they liked to work alone, unless they were having difficulties with the mathematics, and then they preferred to work in a group. In the survey, 48.9% (71 students) chose an option which included ‘by yourself’ for who they usually study with. In the interviews, fourteen students said they preferred to work alone, as one student reported:

I study by myself. It's good that way 'cause I can focus my thoughts. So if I'm working on a problem, and I can focus on it, I'll be quicker to get it. Because when my friends are around talking, I don't know what I am doing. Most of the time, I study individually.

Another student added that "home is always the best place to study, but sometimes we need to group study so we help out with our problems, you know".

Students that were having difficulty with mathematics or feeling insecure about their ability to do mathematics preferred to always work in groups:

I find it comfortable to study in a group because I never know what I'm doing wrong not until my friends tell me that my equations and working out I've done has a flaw in it, so fix it up. This helps me remember when the time comes in the test that this is what my friends told me is wrong so I know the right way to do it.

Another student insisted that "it is easier to study with other people ... when you don't understand something there is someone else you can ask". Another commented: "with a lot of students there seem to be a lot of points of view so we can discuss it more. There will be more questions to ask and when he [the lecturer] discusses it, it will be clearer". The students described groups particularly beneficial when studying for tests or exams to pool their mathematics knowledge and get help:

if you study with the group it is really helpful because some students do the comparison. They both solve one problem and compare ... how come they don't have the same answers so they like work together in order to get that one problem done. So it's really, really helpful to work in a group.

However, working together as a group meant finding somewhere that discussion and noise was permitted. The preferred locations were some large picnic tables in the foyer area under cover, in the fale (meeting house), or in empty classrooms, where they could engage in discussions about their work.

Students will seek help with their mathematics if they feel comfortable

The students described Pacific students in general as shy because "they are very afraid to ask when there are a lot of people in the room ... they are like shy to raise up their hand and ask the teacher to go back and explain something" and that "Pacific kids are shy, they have pride, they don't want to show any weaknesses". Small groups of about five were preferred for working and one student explained that "if you want students to come to class to help them they won't do it. But, if you tell them to come to class so they can get better I think that would help to make them come". Classes were another name for tutorials, which were unanimously regarded as the best place to get help with their mathematics as one student claimed "they explain more what we did in the lecture" and another stated "they get to slow down for us. In the lectures they go at a pace and when we come to tutorials we can ask some questions and can go back".

The students also talked about having good relationships with their lecturers as this made it easier to seek help. One student said: "last year, we weren't comfortable asking the teachers because we didn't know them well, but now we're sort of connected, so it's easy if we need help to just go to the lecturers". Another student added: "we're not afraid to ask as there is a warm relationship between lecturers and students".

Students preferred to learn their mathematics using both English and their first language

Over two-thirds of the students in the survey indicated they use both their first

language and English to discuss mathematics with their friends. They defended this saying it was easier to explain and understand mathematical concepts to each other or with their lecturer. One student elaborated saying

We understand the simple basic words but when it comes to mathematical terms it's very hard to really understand what that really means until someone else says it in the Pacific way. Sometimes it happens to me when the teacher says something that's mathematical and I don't understand. Its either ask the person next to me or I ask the teacher to interpret.

Another student confirmed that "there is a bilingual language [at university], the tutor or the lecturer talks in bilingual; they either speak in the [Pacific] language or English".

The students were comfortable switching between the two languages. One student said "when we study with friends we go back and forth with English and our first language". However, the students also knew that it was important to be able to explain mathematics in English in order "to be able to communicate with other countries" as one student clarified. When asked about how they might get on with language if they were at a university abroad there was a fear it would be too hard to only communicate in English and that they preferred the bilingual method. One student remarked it would be "quite complicated for me to communicate with teachers and as well as other students in English, I mean having my second language English".

Students perceive studying in New Zealand has more advantages but the expectations would be very high

Students who studied in New Zealand were perceived 'lucky' as they had access to better resources, better technology, more options and opportunities:

I'm not really sure of life in New Zealand but all I know is that they've got a better chance and better opportunities than what we have here ... if you go to a New Zealand university and get your degree it will be easier for you to get jobs.

The students believed that studying overseas would provide assurance for a good job. One student assumed New Zealand not only had more options but that jobs were easier to find and secure. He believed that "overseas you go straight to what you're looking for ... it's not like over here; you have to figure out where to go".

Although New Zealand was considered the land of opportunity, not all students wanted to go and in particular leaving the strong family bond. One student said: "If I go there, I will miss my family. Definitely I would be homesick. The lifestyle or the culture would impact. They're the opposite of here. The lifestyle in New Zealand is different from here". Another student was concerned that without their families "no one would tell them to do this and do that. Be good at school. Study hard". Some feared that peer pressure would be much stronger and doubted they could cope. Despite these concerns, they all regarded achieving a scholarship to a New Zealand university would be exciting and they would feel proud to go.

Discussion and Conclusion

The university students reported here were goal-oriented, and possessed a clear understanding of the huge family and parental pressure on them to be successful in their studies. The responses reflected a strong social conscience and entrenched cultural upbringing for students to do their duties and fulfil their obligations. As the majority (if not all) of the students live with family (immediate or extended), their responses

demonstrated a grounded and practical, cultural and social understanding of their responsibilities and obligations as family and community members. Strong and high parental and family expectations of the students were evident by students' references to their parents as being strict, and therefore they felt obligated to deliver. Students evidently regarded parent strictness positively, viewing it as encouragement and motivation to develop and maintain good attitudes towards, and to be successful in, their studies.

Students' responses about how they learn mathematics and their perception of what constitutes an appropriate level of questioning have strong cultural and social underpinnings, suggesting a parallel that resonated with traditional forms of learning in a Pacific setting. For example, at the university, there is a formal 4-tier organisational structure of modes of delivery, established to provide students with ample opportunities to learn mathematics. These tiers of learning opportunities were referred to and clarified further by the students themselves during the interviews. At the first tier (whole class setting) was the traditional lecture in which students themselves acknowledged was not the best place to ask questions for clarification and understanding. The second tier (small groups) were tutorials where there was opportunity to raise questions, collaborate with other students and initiate discussions with the tutor for additional clarifications, understanding and elaborations of lecture material and tutorial exercises. The third tier of delivery (one-on-one consultations) with their tutor and/or lecturer provided further specific and additional assistance at the individual level. The fourth tier (on their own) was their need to study alone either at home and/or the university to stay focussed, think things through, reinforce and consolidate their understanding of mathematics.

The second tier is noteworthy, as at this tier there were both formally organised tutorials and student-initiated peer study groups. The latter were complementary providing an informal mode of learning where the students collaborated further, shared ideas and strategies and learnt mathematically from and with each other. The students actively engaged for mathematical understanding by posing questions, collaborating and sharing ideas and strategies. In tutorials they were able to seek further elaborations and clarifications from the tutor as they felt nurtured, supported, safe and more comfortable in this smaller setting. Involvement in a peer study group or joining an established one when they had difficulties provided benefit from more mathematically capable students as evident from comments in section 4.3.

The parallel resonance with traditional, cultural and church forms of formal and informal ways of learning may be viewed through the 4-tier organisational structure in a village and community setting. As an example, if you are a young untitled man (*taule'ale'a*) learning the art of speech making in anticipation of becoming a *matai* (chief or orator), the first tier in a village/community setting would be the larger group such as the village council including your own social grouping of *taulele'a*. Here, learning of cultural practices and processes in action by the young untitled people and others would be through listening, observing and noting; questioning and discussion by the observers are not allowed nor encouraged. Within the composition of the village council would be a group of chiefs and a group of orators. Serving the village council would be this group of untitled young men (*taulele'a*) who are the 'students' in training. For the *taulele'a*, they would need to learn, practice and develop their skills in making a variety of traditional speeches to suit different contexts. This learning, development and practice of making speeches would take place within their own *taulele'a* group (tier-1) with smaller groupings as needed for more intensive and targeted training and rehearsals (tier-2). Active learning would take place at this second tier within smaller group settings where young people practiced what they have observed, contributed when they

felt confident, sought clarifications, revised and continuously refined their knowledge, skills and understanding of the art and practice of making cultural speeches. Like the university students (at tier-3), individuals could seek out one-on-one consultations with more experienced others or their senior members but ultimately they would need to individually create, practice and rehearse (tier-4) in anticipation of pending assessment and evaluation by the village council and the *taulele'a* group at the first tier.

Another important finding from the study is the existing practice, and value, of bilingualism (use of both English and their first language) in all formal and informal teaching and learning situations. The two languages are used interchangeably by both students and lecturers/tutors. In this way, they are not in hegemonic competition, where one language is considered more important than the other, but complement and elaborate so that the intended meanings of mathematical statements and concepts are conveyed, communicated and better understood. Whilst students accepted that mathematics questions in assessment tasks and examinations would be in English, they preferred and looked for the use of both languages during all learning and teaching situations. Their fear of not having this option if they were studying in an overseas university came through very strongly in students' responses. Our findings further confirm those of Airini et. al's [7] findings about the use and value of students' first language, their culture, customs or way of life to gain a better understanding of words or terms. Related to this finding is the importance and value for students developing 'good and warm relationships' with their lecturers and/or tutors in which they felt their needs and aspirations were understood. The students viewed this as an important first step before they could comfortably venture out and ask for help from a lecturer and/or tutor.

Underpinning all these findings was the students' acknowledgement of the importance of continuous family support and their expectation that it was always there for them, for regular guidance and advice whilst they studied at the university. Their fear of losing this support was evident when students discussed their perceptions of what it might be like to study overseas away from parents and family. Besides providing support, an interesting insight was that many students acknowledged and cited that their parents or close relatives were their role models for choosing to take mathematics.

The paper's second focus was to identify challenges faced by these Pacific university students whilst studying mathematics at university. Because these students live invariably with family (immediate and/or extended) their daily lives consisted of practices (a mix of university, family, cultural, and/or social) related to the appropriate enactment of their different roles and responsibilities as university students, family, church and community members. Their duties and meeting their respective obligations would not necessarily all occur every day. The challenge for them is ultimately managing and prioritising their time wisely between their numerous obligations and responsibilities so that they could also achieve their goals as university students. Finally, their practised and constructed realities over a typical week then became that of fitting in most of their various obligations during and outside of university time.

The survey and interviews did not investigate in-depth the strategies students used in learning and solving problems in mathematics in the context of any particular problem. Thus in this study, we do not have data with which to confirm or dispute the learning approach that Phan and Deo [6, p.380] described as "rehearsing, recalling and memorisation without emphasis on understanding". What our data shows instead, are two insights. First, is that these Pacific university students, actively sought, looked for and expected a better understanding of mathematics concepts received through lectures, by raising and posing questions for further clarifications and elaborations as appropriate, to the lecturer/tutor and each other during tutorials, peer study groups and/or during

one-on-one consultations. The learning strategies therefore that these Pacific students practiced and applied in a university setting actually mirrored cultural learning practices in their communities such as seeking assistance from a small group of more (mathematically) capable others when in doubt but ultimately preferring to study/practice on their own when perfecting their skill or enhancing their understanding. The latter is our study's contribution to a better understanding of Pacific students' ways of learning which is contrary to the much publicised "rote-learning" style or "small group learning" in the current literature. Second, unlike Phan and Deo' university students who attended a South Pacific university in which only English is the language of instruction, the students reported here were studying at their national university with an official bilingual policy with all matters educational. Thus they had the advantage of using two languages complementarily with which to seek and receive clarifications and elaborations for a better understanding. Our findings further confirm those of Airini et. al's [7] findings about the use and value of students' first language, their culture, customs or way of life to gain a better understanding and increase success for students.

Whilst our study focussed on determining how Pacific students at this particular university in the South Pacific learn mathematics, the findings from the data presented, namely, (1) students' strong cultural upbringing with a developed sense of their role, responsibilities and obligations; (2) the 4-tiered learning modes; (3) bilingualism to facilitate elaborations and clarifications for a better understanding; (4) developing good and warm relationships with their lecturer/tutors; and (5) continuous family and parental support through their university studies; are not necessarily mathematics-specific. Instead, the findings encompassed factors related to students' cultural upbringing and their identity, and as such, our findings may be generalisable to other university subjects beyond mathematics. Moreover, learning mathematics (or other subjects) by Pacific students studying in New Zealand universities may be further enhanced by taking into account the main findings (especially findings (2) to (5) from our study).

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Lecturer Change: Espoused Versus Enacted Beliefs About Teaching in Large Undergraduate Mathematics Lectures

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Many lecturers use teacher-centred styles of teaching in large undergraduate mathematics classes, often believing in the effectiveness of such pedagogy. Changing these beliefs about how mathematics should be taught is not a simple process. This study describes the journey of a mathematician as he accepted the challenge to ask students to work interactively on questions in large lectures. The mathematician's espoused and enacted beliefs about lecturing were confronted through a cyclic process of developing questions, testing them in lectures and refining them in collaboration with a research group. The study shows how this process of testing and reflecting on teaching practice can be a useful strategy for effecting change in lecturers' beliefs.

Keywords: beliefs; reflection; teaching practice; undergraduate mathematics education

Introduction

Changing ones' beliefs about lecturing is not easy and often extremely uncomfortable. However, research in tertiary mathematics education suggests that this process can become easier when mathematicians and mathematics educators reflect on their teaching practice collaboratively using appropriate theoretical tools [1]. Paterson, Thomas and Taylor [2] describe an example of a positive collaboration between mathematicians and mathematics educators who analyse video-clips of each other's lecturing; Hannah, Stewart and Thomas [3] report how a mathematician reflected on his teaching decisions in a linear algebra course by sharing journal entries with two mathematics educators. Such forms of collaborative reflection can encourage lecturers to change their espoused beliefs about lecturing. However, we propose that a deeper kind of change occurs when lecturers test out those beliefs in practice.

We describe a case study of Chris (not his real name), a mathematician who changed his beliefs about lecturing over a three-year period in collaboration with a research group of mathematics educators and lecturers. Chris accepted the challenge of asking his students to work interactively on a mathematical question, once during each lecture. This seemingly simple challenge opened up a mine of issues: What kinds of mathematical questions are appropriate? When and how should he ask them? How will students react to this change of lecture norms, from being passive observers to working in small groups? Chris worked with the research group to develop, test, and refine strategies for asking effective questions in lectures. However, these strategies were more than mere instructions – they were also practical instantiations of his evolving beliefs about the role of lectures and lecturers in students' mathematical learning. Thus, by testing out and reflecting on these practical strategies, Chris simultaneously examined and evaluated his beliefs about lecturing.

This paper demonstrates that a lecturer's espoused and enacted beliefs about their practice can be modified through a cyclic process of implementing, testing, reflecting

and revising one's teaching practice. We begin by reviewing the literature on teacher beliefs and teaching practice, and how teacher beliefs can change. Next, we present our conceptual framework that demonstrates how espoused beliefs can be evaluated against enacted beliefs. The story of Chris then follows, detailing his journey of testing and revising his prior beliefs in collaboration with the research group. We end with a discussion on how these findings might inform professional development in tertiary mathematics teaching.

Beliefs and Teaching Practice

Beliefs can be described as personal judgments, intentions, expectations, and values; or more simply, as lenses through which humans view the world [4, 5]. All teachers hold beliefs about their work, their students, their subject matter, and their roles and responsibilities [4]. Beliefs are so powerful they will filter what is seen, and what is seen will affect beliefs. They can persevere against contradictions caused by reason, time, schooling or experience [5, 6].

According to some researchers [7-10] teacher beliefs commonly fall into two categories. The first is the teacher-centred style, where the teacher believes that knowledge is delivered or transmitted, and that frequent testing is necessary to check on progress. In this style, there is little recognition of the value of student errors as part of the learning process, and the teacher is expected to play an authoritative role. The second is the student-centred style, where the teacher believes students play an active, central role in constructing their own mathematical knowledge. The teacher believes in a supportive climate in the classroom with discussion and exploration of problems related to the outside world.

As teachers search new understandings of mathematics and the learning and teaching of it, their position may shift [7, 11]. However, faced with the constraints of actual classroom teaching, teachers may position themselves different to their beliefs. In other words, although teachers espouse certain beliefs of theoretical principles, there is often no evidence these are implemented in their practice. In order to change practice, new beliefs must be created, because old beliefs act as filters and can redefine what has been seen [5, 6]. Individuals tend to hold onto beliefs based on incorrect or incomplete knowledge, even after scientifically correct explanations are presented to them.

The literature identifies three necessary conditions for teacher beliefs to change. First, teachers must acknowledge their current practice is problematic [12, 13]; next, they must have an opportunity to trial new practices [5]; and finally, they need to reflect on their existing mathematical beliefs and knowledge [5]. However, modifying long-held, deeply rooted beliefs and conceptions about mathematics and the teaching and learning of it, over a short period, is not easy [12].

A number of writers describe the third approach, reflection, as the key to changing beliefs [4, 14, 15]. Through reflection, teachers gain an awareness of their implied assumptions, beliefs and views, and become aware of viable alternatives. Although a teacher has the tools to reason, judge, weigh alternatives, reflect, and finally to act, any change will come about from reflecting on their experiences [16]. If change then occurs, their belief system will also undergo change and be restructured.

Conceptual Framework

Our conceptual framework (summarised in figure 1) acknowledges that a lecturer's beliefs can be exhibited in two ways. A lecturer can describe their beliefs on a theoretical level (espoused beliefs), and they can demonstrate their beliefs through their

practice (enacted beliefs). The beliefs exhibited in these two ways may not always be consistent: a lecturer may espouse a set of student-centred beliefs when talking about their lecturing to colleagues, but enact a different set of beliefs consistent with a teacher-centred approach in practice.

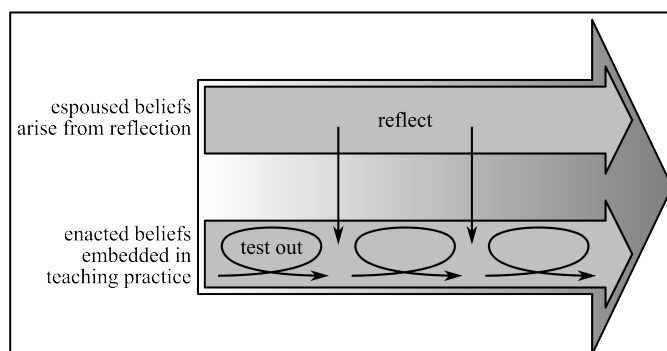


Figure 1. Our conceptual framework for change in lecturers' beliefs.

In our study, we are concerned with changing lecturers' beliefs in a way that ultimately impacts on their practice. It is quite straightforward to convince a lecturer of the importance of student-centred teaching at a theoretical, espoused level. It is harder to change their beliefs at a more fundamental, practical and enacted level so that they also change their practice. We consider it necessary to address the beliefs that lecturers exhibit at both levels when trying to effect lasting lecturer change. In our approach, we challenge lecturers to implement practical teaching strategies that are embedded with teaching philosophies that differ from the beliefs they currently espouse and enact in practice. For example, this paper describes the results of a challenge we gave a lecturer of very large classes (of up to 350 students): to ask students to work interactively on a mathematical question once per lecture. This seemingly simple teaching strategy was embedded with many issues that needed to be resolved about the nature of mathematical learning, the social norms of large lectures, and the role of the lecturer in students' learning. In accepting the challenge, the lecturer was forced to become aware of, evaluate and revise many of his beliefs about lecturing. He grew in his convictions about the importance of student-centred teaching, and this change was reflected not only in the lecturing we witnessed through the project, but also in the lecturing he planned to do after the project had ended.

The large overriding arrowhead in figure 1 indicates that as lecturers engage in the process, they will be forced to reconcile potential discord between the beliefs they espouse and those they enact. If a lecturer becomes convinced about the value of the implemented strategy, the lecturer's espoused and enacted beliefs will evolve and become closely aligned; if the lecturer rejects the new strategy, the lecturer will revert back to his or her old practices, but will be forced to acknowledge the philosophy of teaching that they are aligned with. In either case, the process encourages lecturers to evaluate the consistency of their kinds of beliefs about lecturing.

Method

The data in this paper come from a research project that investigated the social norms of lecturer-posed questions in large undergraduate mathematics lectures. Five mathematics lecturers and mathematics educators worked together over a three-year period to design and test techniques for implementing questions in large lectures. Multiple forms of data were collected to assess the effectiveness of these techniques: interviews, questionnaire responses and journals from students, and interviews and

written reflections from two of the lecturers.

Participants and setting

We focus on Chris, a pure mathematician with 13 years of experience in undergraduate lecturing. Chris was involved in the project as both researcher and lecturer: together with the rest of the research team, he designed questions and questioning techniques, some of which he then tested in his lectures. Chris was widely regarded as an excellent lecturer even before the project began – he consistently received very positive student evaluations of his teaching, and implemented innovative teaching strategies such as Team Based Learning [17] and tablet PC recorded lectures [18].

Chris implemented the questioning techniques in a large first year calculus and linear algebra course at a New Zealand university. This course catered predominantly for students not majoring in mathematics or related disciplines, and had an ethos of delivering a skill set to students who will use mathematics in other participant areas: business and economics, statistics, computer science and the physical sciences. Approximately 800 students enrolled in the course each semester (fewer in the summer semesters), and lectures were delivered in multiple streams with 100-350 students in each stream. During regular semesters, a team of up to eight lecturers would deliver the same content in three or four lecture streams. Each lecturer followed a common lecture schedule, and taught from pre-published series of lecture slides that most students purchased. The lecturing team considered it important to teach in this consistent manner, in order to prepare the cohort as a whole equally for common assignments, tutorials and tests.

Student interviews from the wider study indicate that the students in this course valued and endorsed a passive lecturing style, even while acknowledging that it does little to help them learn within the lecture environment [19].

Data Collection

Chris implemented the questioning techniques in the calculus and linear algebra sections on three occasions – in 2009, 2010 and 2011, and kept journals reflecting on the process. He was interviewed four times either during the semester in which he taught the course, or immediately after the course finished. Interviews were conducted with other members of the research team present, and were audio-taped and transcribed. The interviews were semi-structured, and the research team asked Chris to reflect on the effectiveness of the questions and questioning techniques he implemented as well as his teaching goals and beliefs. At the end of the project, Chris wrote a reflection on how he thought his goals, beliefs, knowledge, and identity, had changed throughout the project as a consequence of implementing the questions and questioning techniques in his lectures.

Data Analysis

The four interview transcripts and the written reflection were analysed in four stages. First, we used open coding [20] to identify and classify recurring concepts in the written reflection. These were refined and categorised into a coding scheme consisting of twelve primary codes (see Table 1) and 39 subcodes.

Table 1: Primary Coding Scheme

Beliefs	Community of Practice	Outside Influences
Change	Didactical Contract	Power
Reflection	Emotion/Attitude	Social Norms
Teaching Practice	Identity	Students

Once the coding scheme was established, we then cycled back and forth between applying this coding scheme independently to the transcripts and reflection, and meeting in groups to compare and revise our coding until consensus was reached. Next, we used axial coding [20] to distil relationships between the codes. Five of the primary codes (beliefs, change, reflection, emotion/attitude, and teaching practice) emerged, with 20 sub-codes (see Table 2), as being tightly interconnected in describing how Chris' beliefs about lecturing changed over the three years.

Table 2: Sub-Codes

Primary code	Sub-codes	Primary code	Sub-codes
Teaching Practice	Interaction	Change	Growth
	Questions		Habituation
	Strategies		Planned Change
	Goal		
	Style		
Emotion/Attitude	Awareness	Reflection	Confidence
	Blame		Discomfort
	Difficulty		Fear
	Evaluation		Fun
	Justification		Pressured
			Surprise
	Tentative		
Beliefs	No sub-codes		

In the final stage, we used pattern coding [21] independently to identify recurrent themes and stories that emerged from the data that were attached to these five primary codes, then compared and revised the themes and stories until we reached consensus. Quotes were then added by relating these back to the original codes. This was then confirmed by Chris as a true account of the events described in this paper.

Results

Chris' beliefs about lecturing at the beginning of the project

Chris was a fairly typical mathematics lecturer before he became involved with the project. He was a confident mathematician and saw his role in lectures as putting forth knowledge. This involved a lot of talking on his part and his style could be described as being primarily transmissive.

Early on in my teaching career, I would have been reluctant to spend more than a small amount of time making my lectures more interactive. I think I gave good lectures, but for the most part they were 'sage on the stage' style.

His students would arrive, sit passively, scribble down essential notes and leave

without him knowing if they understood the concepts being taught. The classes were quiet as interaction was kept to a minimum. The lecturer was in charge. This was a routine that Chris was used to and had been established from his own undergraduate experience. He had not challenged his approach to delivering large lectures as he thought the current situation worked well.

Although Chris was enthusiastic for changes he was aware of the limited time available in lectures to devote to questions and wondered how this would play out. He told us that

Stepping back from the position of control to give students time to talk about mathematics was not something that I was particularly comfortable with. When I did ask questions in lectures, they were often mundane: either “what is the next entry in the matrix” kind of questions – almost rhetorical; or at most procedural questions which reinforced a recent example.

As Chris mentions, any questions that he might ask the class tended to be procedural and closed so as not to waste precious time. He gave us an example of a question he had used, describing it as typically closed and process-driven:

Question: Show that the function $f(x) = x^3 - 6x^2 + 20x - 24$ has no critical points and one point of inflection.

These types of questions reinforced the notion that mathematics is procedural and can be acquired as a skill set or tool box of routines to be applied. When students volunteered answers to questions posed in lectures, Chris saw it as his role to validate (and if necessary reword) student answers, and had not considered other possible avenues for validation. He expressed that

It's very natural to say okay who knows what the answer was, elicit that response from the class, get an answer you know, validate it, rephrase it in your words so you're happy with the answer that everybody copies down and then go on with the lecture.

Chris remarked that he felt constantly under pressure to deliver the pre-published content in the prescribed time and referred to this often as the ‘tyranny of content’:

I had mostly felt the pressure to deliver the ‘expected’ product. The phrase ‘tyranny of content’ comes to mind; there is a pressure to deliver everything in the course material. This is especially true for this course, in which there are multiple streams and pre-prepared lectures. There is a pressure to deliver the same experience that other streams get; skipping lecture slides to spend more time on something else diverges from the status quo.

At each lecture Chris would introduce new topics as they came with comprehensive power point slides to supplement his teaching. Content coverage was an overriding goal, and Chris explained that he would feel guilty if he had not delivered the ‘expected’ content. The students would observe the slides Chris used and watch him execute examples so that they could learn to do these for themselves later on for homework. Chris believed that this was how large lectures should be run, and that his control over the time, content and students reflected his effectiveness as a mathematics lecturer.

Chris’ lecturing beliefs at the end of the project

After joining the research group, Chris was keen to introduce some different teaching strategies that might benefit his students during the lectures. Despite having

concerns, he willingly took on the challenge to ask students to work interactively on questions during his lectures. Over the course of the project, as Chris tested these out and reflected on them, several key beliefs changed for him.

He began to take a more scientific approach to his teaching, whereby he set goals, tested them out, and observed the effects of his implementations. Chris noted that he now had more “confidence in saying I think my question is more useful than that example” and told students that “you don’t learn by watching me do it”. As Chris asked students to work on more questions and reflected on the process each time he kept an account of what worked, what didn’t, and what changes he could make.

Another change for Chris is the amount of talking he would do in a lecture. Instead of talking for most of the lecture at the students, Chris now facilitates more class discussion times. This involves using open conceptual questions to get students engaged in thinking about and discussing the mathematics, first with their neighbours. Chris illustrated this change with an example he professed was one of his most effective questions:

Question: A student is trying to find $\lim_{x \rightarrow \infty} \frac{f(x)}{x}$ and writes the following:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} f(x) = \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \lim_{x \rightarrow \infty} f(x) = 0 \cdot \lim_{x \rightarrow \infty} f(x) = 0$$

Is this correct?

In the implementation of questions such as the one above, Chris used a framework of ‘Stop, Think, and Discuss’ suggested by the research team. After the question was presented the lecture was stopped, and students were instructed to work alone on the question and then to discuss it with the others around them. The size of these groups varied. Chris used this time to move around the room and interact with the students and take note of some groups’ answers. He then asked the class to feedback short verbal answers, or he summarised the common answers he had heard arise. This type of question encouraged students to discuss the underlying mathematics: “They’re getting something out of it that they can’t get out of the question by reading it in the textbook or doing it at home by themselves.” Chris would regularly tell the research group “I want them to be discussing the maths” and that he now believed that this was important for the students’ learning. As a bonus, he found that he also “enjoyed it [the lecture]. It’s nice to be giving them the opportunity to come up with the ideas themselves to see some of the concepts coming out of the examples”.

The students, Chris said, had become used to the ‘stop, think and discuss’ sessions and would actively engage in the questions. Chris could see that this was the result of the type of questions he was asking: “I think really a lot of the stop and think kind of thing is coming down to the question design being in this more kind of open ended style”.

Chris’s attitude to lectures has changed, and he no longer labours under the ‘tyranny of content’. He believes that each lecture should focus on the underlying concept, and prioritises these such that “the goal is for them to understand the big ideas of the course”. He espouses a new belief that procedural questions which reinforce mathematical skills are of limited use in lectures, and there is evidence that he is beginning to enact this belief in his teaching practice. The change has been gradual and ongoing. Chris is aware that for his students:

They’ve got an opportunity to think about what you’ve just taught them, what you’ve

been talking about at least, to see whether they've learnt anything. So I think that it kind of, yeah, it gets their brain working on a different level. It gets them thinking about the material rather than just sitting and passively listening to it.

Chris has continued to reflect on his lectures and he feels more confident about his future as a teacher.

The process whereby Chris changed his beliefs

Chris's beliefs changed through regular reflection on his teaching practice as he introduced different questions in each lecture. He commented that "I feel more with each semester that this is the right thing to be doing for students in this course". As he critically tested the questions out, he set goals and made modifications to his previous teaching routines. Working within a research team as part of a community of practice provided insightful observations and suggestions to develop his teaching practice and make it more effective. The process of reflecting on the implementation of the questions was particularly effective in bringing about some of the changes in his beliefs, which he would often talk about with the group in research meetings.

The first phase of change for Chris was a period of setting goals, associated with some expressed trepidation: "You don't know where you're going to go ... It does move you out of your comfort zone the first few times". He could foresee implementation barriers for his questions, in terms of engaging traditionally passive students, the tyranny of content and the difficulty in validating student answers. Chris commented that

you start out as a new lecturer and kind of feel like it's your job to be down at the front putting forth knowledge to the students to absorb and so [if the students are] spending five minutes or ten minutes where they're talking to each other but you're not talking to them, how is that giving a lecture?

However, Chris continued with cycles of implementation and reflection, and his fears diminished over time.

The second phase of change for Chris was a period of goal setting, and putting these goals into practice. Chris set out to change the implementation of his questions in the lectures to be a more integral part of the course, and embraced the importance of being a facilitator in lectures, allowing time for students to discuss the mathematics at hand. He often expressed surprise at the effectiveness of this change saying "it was quite unexpected that there would be that level of engagement and discussion". Because of this unanticipated success, Chris was surprised that some barriers to student engagement persisted. However, he knew the process was "still a learning experience, coming up with the question that gets them to engage with the mathematics beyond just monkey see, monkey do".

The final phase of change for Chris involved a period of reflection and confidence. Critical evaluation of his teaching practice became a regular focus of discussions within the community of practice. The group could see a growing confidence in Chris's beliefs about the nature of effective teaching in large lectures, and the methods he uses to implement them. The fear he had often expressed at the start about the prospect of asking students to work on questions during lectures was mostly gone.

Discussion and Conclusion

Did Chris' beliefs change? His espoused and enacted beliefs were initially at odds, but over time the gap between them decreased. In later interviews Chris described his

beliefs and how he acted on them in his lectures: by encouraging students to think more about mathematics in lectures; by using open-ended questions for class discussion; and by rejecting the tyranny of content. The change in his enacted beliefs was seen against a backdrop of changing emotions. After repeated testing out and reflection, Chris developed a confidence to apply the new teaching practice and enact the espoused beliefs. While Chris was at first nervous about changing his teaching practice, surprise followed when he found the new teaching practice effective and enjoyable, or affirming of an espoused belief.

As well as being the subject of this study, Chris was also an active member of the research team. His dual role therefore introduces potential for bias in the results, as Chris was involved in analysing the transcripts and may have been inclined to show himself in a positive light. This was mitigated by the involvement of the rest of the research team, who could dispassionately challenge Chris' perspective. It was advantageous having Chris involved in the analysis, as he was able to clarify aspects where the transcripts were unclear or ambiguous. We are confident that our results have not been skewed by the subject's perspectives, as using we engaged in the coding process individually and independently and were able to reach consensus on the story we saw emerging from the data.

This study also highlights the benefits of Chris' involvement in a research team. Other lecturers would also benefit from testing and reflecting on modifications of their teaching practice with mathematics educators. As such, our results are applicable as a facet of professional development. Working within a team of educators can be a source of encouragement for a lecturer to engage scientifically with his teaching beliefs. In promoting this, we note that lecturers are likely to be apprehensive when implementing new teaching practices, so it is imperative that the team within which to reflect is supportive and a significant period of time to implement a programme of planned change is given.

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Analysis of Foundation Year Mathematics Curricula: Comparing International Trends With King Fahd University of Petroleum & Minerals

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The call for academic standardization and quality control has led to an increasing number of foundation year programs (also called *preparatory* or *transition* programs) around the globe. Such programs act as buffer between two levels of education (typically secondary and tertiary education). The Preparatory Year Program of King Fahd University of Petroleum & Minerals (KFUPM) is an example of a large scale foundation program that prepares the entire intake of the university for undergraduate studies. This paper analyzes the preparatory year mathematics curriculum of KFUPM in comparison with other internationally recognized programs of a similar level. The aim is to identify the strengths and weaknesses of the Program with reference to international best practices of foundation year mathematics. The sampled programs have been selected from different countries to ensure a global perspective. Therefore, the study not only leads to concrete recommendations for improving the program at KFUPM, but is also one of the first documented attempts of comparing foundation year mathematics curricula from different parts of the world.

Keywords: transition program; college algebra; trigonometry; foundation year; preparatory year program; curriculum; curriculum revision

Introduction and Motivation

In the current age of globalization, universities are catering to a broader audience with varied demographical and academic backgrounds. Standardizing the knowledge and skills of students before embarking on undergraduate studies is becoming ever more essential, yet difficult. Resultantly, it has become increasingly important to develop effective transition programs to ensure a smooth transition from secondary to tertiary education. Based on content, such programs can be largely classified into two categories: *specialized programs* that cover the prerequisites of degree courses in a particular discipline and *generalized programs* preparing a large body of students for tertiary education in a variety of subject areas.

The Preparatory Year Program (henceforth referred to as PYP) of King Fahd University of Petroleum & Minerals falls in the latter category. Like most of the contemporary foundation programs worldwide, mathematics is a core component of the PYP. The PYP offers two coordinated mathematics courses, namely Math001 and Math002⁵, which are essentially college algebra and trigonometry courses. Each one is a four credit hour course and runs over a semester.

Studies have shown that students' performance in the PYP has a lasting impact on their future educational endeavors [1] in general, and their performance in higher mathematics courses [2] in particular. Therefore, to maintain the quality of KFUPM

⁵ For course descriptions and syllabi refer to www.kfupm.edu.sa/pypmath

graduates, it is imperative to look critically into the PYP mathematics curriculum and determine where it stands on the scale of similar international programs. Furthermore, despite the growing literature on the impact of foundation year programs in different settings (see [3, 4]) very little is known regarding the programs in the Middle East. Additionally, to the best of our knowledge no comparative analysis of foundation year mathematics curricula has been carried out yet. This paper aims to fill this gap. Although we use the PYP mathematics curriculum of King Fahd University and Petroleum & Minerals as our reference point, our work provides a comprehensive account of the international practice of foundation year mathematics.

Literature Review

In recent years, university foundation year mathematics has attracted the interest of many researchers. This is partly because of the increasingly diverse mathematical backgrounds of the students entering tertiary education, with a significant percentage under-prepared for university mathematics [5]. Another factor contributing to this growing interest is the emphasis on academic standardization. In particular, some of the areas that have been under discussion include students assessment at the foundation year level [6], development of curriculum and pedagogy to better address students needs [7], a theoretical model for high school to university mathematics transition [8], the use technology in the teaching and learning of mathematics at tertiary level [9], special related issues [10] and student motivation [11].

Although some studies aim at measuring the effectiveness of foundation year mathematics curriculum in achieving its objectives [11, 12], these studies mostly focus on a particular program and try to determine the usefulness of the curriculum based on student perceptions and achievements. So far, to our knowledge, there has been no documented attempt to compare foundation year mathematics curricula of two or more foundation programs in any context.

Considering the foundation year programs in the Middle East, there has been some research centered at specific program aspects [2, 13]. The lack of English proficiency, which is especially relevant for bilingual Arab students in the Middle East, has also been a topic of active debate [14, 15]. However, no studies have been undertaken to examine the foundation year mathematics curricula prevalent in the region.

Methodology

The task of comparing mathematics curricula across programs is technical to say the least. Even a tangible measure like student achievement is not universally accepted as a good indicator of the success of mathematics curriculum and can generate contrasting views [16, 17]. We therefore avoid any attempts of weighing up the merits and demerits of different foundation year mathematics curricula. We focus instead on determining the extent to which the mathematics curriculum of the PYP covers the topics most commonly taught by other internationally reputed programs of a similar level. Our approach consists of three major phases:

Defining the selection criteria

The first step taken was to come up with a set of guiding principles for short-listing the programs to be selected as samples. This was done with the aim of developing a list that adequately represented international best practices of secondary to tertiary transition in mathematics. Moreover, the selected programs have similar objectives, level, medium of instruction and mathematical content as the PYP mathematics courses

to ensure a meaningful comparison. Therefore, we based the selection on seven critical variables, namely, program objectives, medium of instruction, level of the program, mathematical content, regional and international reputes, achieving significant sample size and global coverage.

Program selection

After exhaustive search, with the above criteria in mind, the following programs were eventually selected as sample for this study. Table 1 lists program titles together with the abbreviations that will be use in the sequel.

Table 1. Short-Listed Programs

Region	Program Title
North America (36%)	Iowa State University Mathematics Remedial Courses (ISU) Kansas State University Service Courses (KSU) Louisiana State University Remedial Math Courses (LSU) Ryerson University Canada (RU)
United Kingdom (27%)	University of Bristol Foundation Year Program (UB) University of Leeds International Foundation Year (UL) University of Manchester Foundation Year Program (UM)
Middle East (27%)	Saudi Armaco Orientation Program (ARAMCO) Qatar University (QU) United Arab Emirates University (UGRU)
Australia (10%)	University of Queensland Foundation Year (UQL)

Data collection

Despite the spread of information through the internet, gathering information regarding foundation year mathematics curricula was not found to be easy. Although, most of the foundation program homepages contain a large amount of data, the information is generally unstructured and informal. In fact some programs do not make much information available through the web. Therefore, questionnaires were sent out to some program coordinators requesting specific pieces of information.

Comparative analysis

The analysis has been carried out focusing on the PYP mathematics courses. The courses have been compared with the courses in the short-listed programs to point out the discrepancies between the two. We basically tried to answer the following questions:

1. What is the demographical and mathematical background of the students attending the sampled programs?
2. What are the prerequisites of the courses offered?
3. Do the programs offer courses of different levels via different possible tracks for the students? For instance do they offer fast-track, part-time or online routes?

4. How favorably the mathematical content of the PYP courses compares with these programs?

Findings

The findings of this study can be categorized into four parts. The following sub-sections elaborate on each.

Student Background

The academic background of students attending the sampled programs is quite similar to the PYP mathematics students (i.e., 12 year high school education). However, there is a significant difference in the linguistic backgrounds. A large number of students attending the foundation mathematics courses in the United States are local residents and native English speakers. The situation is different in United Kingdom, Australia and Canada where a vast majority of the foundation year students come from overseas. Therefore, most of the students attending these programs are in a similar situation as the students in the Middle East who learn English as their second language.

Prerequisites

All the sampled programs require entering students to have 12 year high school education with high school mathematical background. In the United States, however, the intermediate algebra courses admit students with middle school mathematical background as these courses are followed by more intensive college algebra courses. In United Kingdom, University of Bristol admits students without A-levels mathematics background to its introductory foundation courses (MATH 10500 and MATH10600)⁶.

Flexibility and customization

One of the major disparities between the teaching of mathematics courses in the PYP and the sampled programs is the greater flexibility offered by the latter. Students are classified in the appropriate level according to their mathematical background and/or scores on the placement test. Some programs – for example ISU, RU, UL, UM and QU – offer different courses according to the proposed area that a students intends to specialize. This is in sharp contrast with the PYP where all the students attend both Math001 and Math002, unless they get an exemption based on promotion exam results. And this is regardless of what they want to study at the degree level.

A distinctive feature of all the programs sampled from the United States is the availability of multiple tracks. Some programs implement innovative measures to facilitate students' understanding of mathematics. For instance, KSU offers Studio College Algebra in addition to the traditional college algebra course. The studio approach is centered at demonstrating mathematics in a lab environment to reinforce learning. The R2R program in LSU provides students with a 'learning lab' staffed with qualified teaching assistants and instructors.

⁶ The complete list of foundation year mathematics courses offered by the selected programs is available from http://faculty.kfupm.edu.sa/pyp/malikhan/sampled_courses.pdf

Table 2. Alternative Course Routes

Program	Tracks
ISU	Self-paced/extended educational route, College algebra with visualization
KSU	Studio College Algebra
LSU	Fast-track, R2R
UM	Fast-track
UGRU	Math Fast Forward Program
UQL	Standard (1 year) and Express (6-9 months) programs

Curriculum coverage

The analysis of the material taught in the short-listed foundation programs reveals some interesting trends. In the following we compare the mathematical coverage of the selected programs with that of the PYP. The findings have been presented region wise as follows.

North America:

Since calculus is introduced during the undergraduate studies in the American educational system, the foundation courses from United States do not cover calculus topics. Instead the emphasis is placed on college algebra and trigonometry. Some universities in North America offer special *Business Mathematics* courses that cover basic topics in probability, statistics and linear programming. Among topics not included in the PYP curriculum, modeling and applications involving exponential and logarithmic functions are taught in all the sampled curricula. Combinatorial topics are covered by some programs. On the other hand, among topics taught by the PYP mathematics courses, *vectors* and *conic sections* (with the exception of *parabola*) are generally not covered by the North American programs under discussion (see Table 3).

Table 3. Comparing the PYP With the Remedial Courses in the North American Universities

Program	Program Topics not covered at the PYP	PYP topics not covered by the Program	PYP topics given less weight by the Program
SU	Sequences and series; polar coordinates; parametric equations of conic sections; mathematics of finance; linear programming; introduction to probability and statistics; rate of change of functions.	Review of basic concepts (such as sets, factoring, fractions, radicals and absolute value); vectors.	None
KSU	Geometry (area and perimeter of circles, the Pythagorean theorem); applications of exponential and logarithmic functions; partial fractions; laws of sines and cosines with applications.	Conic sections; vectors.	None
LSU	Exponential growth and decay; compound interest and APR; determine the future value of money.	Matrices and determinants; conic sections; vectors.	None

RU	Probability and statistics for business; calculus; vectors in space.	Conic sections	Trigonometry; graphing polynomial functions of higher degree; graphing rational functions.
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United Kingdom

Due to the early introduction of calculus in the British educational system (typically during A-levels⁷), the foundation year programs in United Kingdom place a lot of stress on calculus topics, sometimes going as far as elementary ordinary and partial differential equations. Other areas that are given substantially more importance compared to the PYP include linear algebra, mechanics, combinatorics, probability and statistics. Overall, as Table 4 suggests, the courses are more intensive and broad-range compared to the PYP mathematics courses.

Table 4. The PYP Versus the British Foundation Year Programs

Program	Program Topics not covered at the PYP	PYP topics not covered by the Program	PYP topics given less weight by the Program
UB	Exponential growth and decay; compound interest and APR; sequences and series; Euler formula of a complex number; Argand diagram; probability and statistics; square matrix as a transformation in the plane or space; introduction to eigenvalues; calculus.	Sets, solving radical equations; graphing polynomials with degree higher than 2; graphing rational functions; exponential and logarithmic equations; conic sections (hyperbola and ellipse); systems of non-linear equations.	Radical expressions; absolute value; graphs of exponential and logarithmic functions; graphing trigonometric functions; properties of logarithmic functions.
UL	Binomial theorem; Pascal's triangle; sine and cosine rules; vector product; mechanics; kinematics and kinetics; probability and statistics; linear difference equations; eigenvalues of a matrix; linear programming; calculus; vector analysis.	Sets, basic arithmetic (factoring, fractions, radicals, absolute value); complex numbers; solving radical equations; graphing polynomials with degree higher than 2; graphing rational functions; algebra of functions; remainder and factor theorem; exponential and logarithmic equations; conic sections (hyperbola and ellipse); systems of non-linear equations.	Graphs of exponential and logarithmic functions; graphing trigonometric functions; properties of logarithmic functions; inverse function.

⁷ The A-level courses are generally taken after completing secondary school and form years 12-13 of British educational system.

UM	Partial fractions; arithmetic, geometric and binomial series; De Morgan's Laws; elementary logic; proof by contradiction; proof by induction; complex numbers: modulus, and argument; algebraic operations in polar form; Argand diagram; calculus; probability and statistics; vector analysis; dynamics; statics; numerical methods.	Conic sections.	None
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Middle East:

While all the programs short-listed from this region are focused on the teaching college algebra and trigonometry, ARAMCO is the only program that offers calculus courses. A comparison (see Table 5) of the college algebra and trigonometry courses from the sampled programs with the PYP courses reveals that the latter offer greater depth and breadth in college algebra and trigonometry, covering several topics that are not taught at QU and UGRU.

Table 5. Comparison With Programs in the Middle East

Program	Program Topics not covered at the PYP	PYP topics not covered by the Program	PYP topics given less weight by the Program
ARAMCO	Calculus; probability and statistics; counting, permutations and combinations; <i>Mathematica</i> ; business mathematics.	None	None
QU	A lot of time and attention is focused on understanding and practicing basic arithmetic.	Zeros and graphs of polynomial functions of higher power; graphs of rational functions; matrices and determinants; conic sections; vectors; inverse trigonometric functions and equations.	None
UGRU	Probability and statistics, applications of exponential and logarithmic functions	Remainder and factor theorems; zeros and graphs of polynomial functions of higher power; graphs of rational functions; matrices and determinants; conic sections; vectors; trigonometric equations; inverse trigonometric functions and equations.	None

Australia:

Unlike the other sampled programs, University of Queensland only offers one foundation mathematics course. Due to the comprehensive and relatively advanced nature of this course, topics such as *sets*, *basic arithmetic*, *trigonometry*, *conic sections* and *vectors* are not covered at all. Instead *calculus*, *probability*, *statistics* and *combinatorial mathematics* are included.

Table 6. Comparison With UQL

Program	Program Topics not covered at the PYP	PYP topics not covered by the Program	PYP topics given less weight by the Program
UQL	Arithmetic and geometric sequences; binomial theorem; probability and statistics; calculus; applications of exponential and logarithmic functions	Sets; basic arithmetic; trigonometry; conic sections; vectors.	None

Suggestions and Outcomes

Several lessons can be learnt from other foundation year programs. If we limit our attention to the treatment of college algebra and trigonometry, the PYP courses can compete with most of the selected programs based on the depth and breadth of the coverage. Trigonometry, graphing techniques and matrix theory are especially well represented in the curriculum. However, this essentially makes the PYP courses calculus-driven. Some important areas such as combinatorics, basic probability and statistics are missing from the PYP curricula. It can be argued that students will encounter these topics later on during undergraduate studies. However, the handling of the material in undergraduate courses is more advanced and this can potentially leave a gap in some students' understanding.

Based on these findings we recommend the following changes to be considered for the curricula of the PYP mathematics courses at KFUPM.

Table 7. Proposed Curriculum Changes

Topics	Recommendation	Precedent/Justification
Conic Sections	Drop	All the sampled programs except ISU and ARAMCO.
Vectors	Drop	ISU, KSU, LSU, QU, UGRU, UQL,
Graphs of Rational Functions	Drop	UB, UL, QU, UGRU, covered by calculus courses.
Polynomial Functions of Higher Degree	Reduce	UB, UL, QU, UGRU
Basic Counting and Probability	Include	All sampled programs except QU and LSU
Applications of Exponential and Log Functions	Include	KSU, LSU, UB, UGRU, UQL

The suggestions in Table 7 are based on the premise of having a single track

preparatory year program. However, we believe it is necessary to move towards customized curricula tailored for individual student needs. As a first step, we need to treat high and low achieving students differently. Offering remedial, traditional and fast-track options for weak, average and high achieving students respectively would make the PYP courses more accessible and enjoyable for the audience.

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University Lecturers' Views on the Transition from Secondary to Tertiary Education in Mathematics: An International Survey

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This paper deals with a very practical issue. In many countries there is a gap between school and university mathematics. The transition period from school to university can be hard for many students. Even students with good marks in school mathematics experience difficulties at university and sometimes fail the first year university mathematics courses. Different parties – school teachers, university lecturers, first year university students, administrators, researchers – might have different views on the reasons for the gap and the ways to narrow or fill it. The purpose of this paper is to present and analyse responses of university lecturers worldwide to a short survey concerning the transition period between the school and university mathematics.

Keywords: transition; school; university; lecturers

Introduction

This paper addresses the issue of the transition from school to tertiary study in mathematics, one which is of great concern and the subject of considerable attention around the world. It deals with perspectives of university lecturers from 24 countries. Many university lecturers worldwide feel that there is a gap between the school and university mathematics and there is a need to investigate the ways of reducing the gap. A serious concern was expressed in the report *Tackling the Mathematics Problem* commissioned by the London Mathematical Society (LMS, 1995): “There is unprecedented concern amongst mathematicians, scientists and engineers in higher education about the mathematical preparedness of new undergraduates....The serious problems perceived by those in higher education are:

1. a serious lack of essential technical facility – the ability to undertake numerical and algebraic calculation with fluency and accuracy;
2. a marked decline in analytical powers when faced with simple problems requiring more than one step;
3. a changed perception of what mathematics is – in particular of the essential place within it of precision and proof.

This is no way restricted to those ‘new undergraduates’ who ten years ago would not have proceeded to higher education. The problem is more serious; it is not just the case that some students are less well-prepared, but many ‘high-attaining’ students are seriously lacking in fundamental notions of the subject.”

Many researchers writing on the transition period from school to university education in mathematics also indicate mathematical under-preparedness of students entering university (Luk, 2005; Kajander & Lovric, 2005; Guzman et al., 1998; Leviatan, 2004; Hourigan & O’Donoghue, 2007; Barnard, 2003; Selden, 2005). They provide a number of reasons for that under-preparedness (a recent trend of moving from elite to mass university education, lowering the mathematics standards at school and

university, inadequate funding, etc.). Research has shown that mathematics students from UK (Hoyles, Newman & Noss, 2001), Hong Kong (Luk, 2005) and Ireland (Hourigan & O'Donoghue, 2007) tend to adopt a surface learning approach in schools but are expected to apply deep learning in tertiary mathematics. The Irish situation, according to a Chief Examiner's report of Leaving Certificate examinations (Hourigan & O'Donoghue, 2007) highlights the problem of poor relational understanding. Based on case studies of two schools, Hourigan and O'Donoghue (2007) found that the examination-oriented nature of the educational system tends to promote a faster pace of teaching, routine mastery of algebraic procedure and 'learned helplessness'. Consequently, surface learning is seen as a quick fix in the schools and creates a culture of learning that fails to prepare students for tertiary level Mathematics (Hourigan & O'Donoghue, 2007). Brandell, Hemmi and Thunberg (2008) provide a good overview of the situation in Sweden. The present Swedish national curriculum came into effect in 1994; thus the stability of the curriculum allows researchers to focus on issues such as the comparison between the "goals and ambitions of mathematics education in Swedish upper secondary school with the expectations of the new students held by the tertiary level" (p. 39). In one Swedish study a questionnaire to secondary teachers asked them to grade how well prepared they thought their students would be to tackle a variety of typical problems from a tertiary transition programmes preparatory course. Specific questions were found to be outside the curriculum and for others the teachers believed that even the better students lacked the necessary concepts and skills to "make sense of the exercise" (p. 41). A study at a New Zealand university by James, Montelle and Williams (2008) analysed students' performance over the years when moving from school to university. Their analysis though led to many more questions. "Have changes in assessment in fact affected the ways in which New Zealand teachers deliver their material, or has it only required some minor changes and adjustments in their curriculum? Is this a phenomenon just applicable to mathematics? After all, mathematics is a cumulative discipline that may be entirely suitable for modularization, unlike some other disciplines. Furthermore, is this initial study too early for full effects to be recognizable? And for those favourably disposed towards modularization, is the 'status quo' result disappointing to the supporters and instigators of the new qualifications? Would they have preferred to see rather a marked increase in the abilities and capabilities of the prospective tertiary level student, rather than a maintaining of standards?" (p. 1048). While much can be learned by teachers in the secondary school and tertiary sectors from studies that recognise the existence of curriculum and other gaps in transition, this is only the first step. The next step must surely be to analyse its causes and then try to do something about it. A widening gap appears to be a worldwide phenomenon and in many countries there is concern that differences in emphasis between school and tertiary mathematics may be increasing and several authors emphasise the importance of the issue of transition from school to university mathematics for students' success in university mathematics (Crawford et al., 1994; Gusman et al., 1998; Anthony, 2000).

Research Frameworks

In this study, practice was selected as the basis for the research framework and, it was decided "to follow conventional wisdom as understood by the people who are stakeholders in the practice" (Zevenbergen & Begg, 1999). The idea of this study has arisen from and is based on teaching practice. This study is primarily a practice-based research study with the aim of identifying and promoting pedagogical strategies that may make the transition period smoother and more beneficial in terms of learning. It is

the teaching/research nexus.

The theoretical framework of the study is based on Piaget (1985) concept of cognitive conflict and David Tall's (1991, 1997, 2004a, b) works on advanced mathematical thinking. One possible reason for transition problems is that there may be both process and conceptual differences in the approaches used at the secondary and tertiary level. A developing theory by Tall (2004a, b) suggests that mathematical thinking exists in three *worlds*, the embodied, symbolic and formal. The embodied is where we make use of physical attributes of concepts, combined with our sensual experiences to build mental conceptions. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, where it is possible to switch from processes to *do* mathematics, to concepts to *think* about mathematics. The formal world is where properties of objects are formalised as axioms, and learning comprises the building and proving of theorems by logical deduction from these axioms.

It is hypothesised that the main reason for the gap between the school and university mathematics would be the difference in thinking. Many students are exposed to a formal deductive approach in mathematics for the first time only upon entry to university and may therefore experience a significant amount of cognitive conflict in their first year. "At school the accent is on computations and manipulation of symbols to 'get an answer', using graphs to provide imagery to suggest properties. At university there is a bifurcation between technical mathematics that follows this style (with increasingly sophisticated techniques) and formal mathematics, which seeks to place the theory on a systematic, axiomatic basis" (Tall, 1997). "During the difficult transition from pre-formal mathematics to a more formal understanding of mathematical processes there is a genuine need to help students gain insight into the concepts" (Tall, 1991). It is not an easy task and requires transition from one stage to another in the Piagetian stage theory where the previous knowledge conflicts with new ideas (Piaget, 1985). "The formal presentation of material to students in university mathematics courses – including mathematics majors, but even more for those who take mathematics as a service subject – involves conceptual obstacles that make the pathway very difficult for them to travel successfully. And the changes in technology, that render routine tasks less needful of labour, suggest that the time for turning out students whose major achievement is in reproducing algorithms in appropriate circumstances is fast passing and such an approach needs to move to one which attempts to develop much more productive thinking" (Tall, 1991). A number of research papers related to the transition period support this claim (Luk, 2005; Barnard, 2003; Hourigan & O'Donoghue, 2007; Selden, 2005; Kajander & Lovric, 2005; Clark & Lovric, 2008, 2009; Hong et al, 2009).

Methodology

Our aim was to present and systematise the responses of university lecturers from different countries to a short questionnaire about the transition from school to university education in mathematics. A cross-country approach was chosen to reduce the differences in cultures, curricula and education systems. The lecturers were surveyed using a combination of two non-probability sampling methods - judgement and convenience. The results of the survey can be treated as a pilot study. The questionnaire was sent to selected participants of international conferences on mathematics education who either teach university mathematics or write papers on mathematics education at university level or both. The response rate was 36% (63 lecturers from 24 countries). The questionnaire comprised of three open ended questions: about the reasons for the gap, the successful remedies that work at their universities and about ideas on what else

can be done to narrow the gap.

We summarised the responses by categorising the participants' answers to the questionnaire and calculating percentages. We also presented the most common answers, strategies and ideas expressed by the participants of the study and added our own comments. We believe that such an exchange of good strategies that work at some universities and suggested ideas that are worth trying can make readers implement them at their own institutions.

The Study

The questionnaire

The questionnaire given to the university lecturers consisted of the following 3 questions:

Question 1. What do you think are the reasons for the gap between the school and university mathematics?

Question 2. What is your Department doing to reduce the gap?

Question 3. In your opinion what else can be done to make the transition period smoother?

The participants' responses

Below we report on the most common responses to the questionnaire. The percentage of participants that identified a similar response (broadly speaking) is indicated. Each participant was randomly assigned a number between 1 and 63, and this number is used to identify a participant with his or her response.

Question 1. What do you think are the reasons for the gap between the school and university mathematics?

Table 1. Reasons for the Gap

Reasons for the gap	Percentages
Higher level of thinking at university mathematics	72
Emphasis on passing the exam at school	37
School syllabus is too broad	34
Too optimistic assumptions and expectations of university lecturers	33
Different ways of teaching/learning	30
Lack of mathematics background of mathematics teachers in schools and lack of teaching skills of university lecturers	26
Lack of communication between school and university	17
Changes in environment	15

Reason 1. Higher level of thinking at university mathematics (72%)

Different emphasis: on calculations, techniques, algorithms, manipulations at school versus on theory, proof, conceptual understanding at university. This difference is reflected in textbooks and assessment.

The teaching style in schools encourages rote learning of disjointed 'facts' and algorithms, which are not underpinned by understanding of the meaning of them or of the fundamental relationship between them. (20)

They learn maths almost without theoretical explanation and only calculation in high

school days.(18)

The school mathematics is aimed to coach for a formal solution of as many exercises as possible, with only superficial understanding the theory, under the everyday instructor supervision. On the contrary, university mathematics is aimed to give in-depth theoretical understanding through the formal delivering lectures with the minimal instructor supervision.(15)

High school math is very mechanical and situational. University math is more theoretical and eventually becomes proof-oriented. The material is qualitatively different and we expect more out of the students. (19)

Most of our students have not seen formal proof before entering university.(22)

There is a major jump in thinking level into the abstract world of proofs. (39)

...'recipes' for doing standard problems. The result of this is that many students don't *have* to understand the ideas behind the problems, just *do* them, and others don't even realise that there is more understanding to have. Being able to perform the right steps is what maths is about.”(51)

To illustrate the above concerns we give two real examples from final school year mathematics exams (university entrance).

Example 1.

Show that the equation $x^2 - \sqrt{x} - 1 = 0$ has a solution between $x = 1$ and $x = 2$.

The model solution given to the markers of the exam reads: “If $f(x) = x^2 - \sqrt{x} - 1$ then $f(1) = -1 < 0$ and $f(2) = 1.58 > 0$. So graph of f crosses x -axis between 1 and 2.”

The suggested solution is based on the special case of the Intermediate Value Theorem which has 2 conditions: the continuity of $f(x)$ on $[a, b]$ and the condition $f(a) \times f(b) < 0$. But only the second condition is checked and the first is ignored as if it was not essential. It was a written exam and all working was required to be shown. The fact that the condition of continuity of the function $f(x)$ was not required by the examiners to get full marks for this question was very disappointing. The message is clear – the manipulations are important but the properties of functions are not. No wonder that students don't pay attention to all conditions of the theorem and properties of the functions – it is simply not required.

Example 2.

Solve the equation $\log_2(9x-1) - \log_2(x+2) = 3$.

Shown below is the model solution given to examiners:

$$\log_2 \frac{9x-1}{x+2} = 3$$

$$\frac{9x-1}{x+2} = 2^3$$

$$9x-1 = 8(x+2)$$

$$x = 17.$$

According to this solution, a check of the validity of the answer seems not to be essential. But ignoring the domain of the logarithm function may lead to the wrong answer as further illustrated by the following example:

$$\log_2(9x - 10) - \log_2(2x - 3) = 0$$

$$\log_2 \frac{9x - 10}{2x - 3} = 0$$

$$\frac{9x - 10}{2x - 3} = 1$$

$$9x - 10 = 2x - 3$$

$$x = 1.$$

It was a written exam and all working was required to be shown. Again the message to students was clear – you can get full marks for a question if you only know how to perform calculations.

The above examples illustrate another concern that our respondents had (and discussed later) – lack of communication between school and university, particularly in setting school mathematics exams. It is hard to imagine a university lecturer who would accept the model solutions for Examples 1 and 2 above as complete solutions for which the student would receive full mark.

Reason 2. Emphasis on passing the exam at school (37%)

“Assessment culture at school means that many students do not learn to understand, they learn to pass exams.” (21)

“School mathematics is not truly 'learned' and stored in long term memory but is quickly lost after the final examination is safely passed.” (20)

Some participants expressed concerns that the mathematics education at the school level is more like training or drilling for certain skills and procedures. The following interesting fact illustrates the point. In New Zealand every three years there is a notable drop followed by a two year increase in the students' performance at the final school year mathematics exam. It was reported by a chief school mathematics examiner that the reason for the regular drops was the change of a chief school examiner every three years. A new chief examiner used their own language style of setting up exam questions which was different from the wording used by their predecessor. The students are sensitive to the wording of the examination questions and it is reflected in their performance.

Reason 3. School syllabus is too broad (34%)

“The high school curriculum is too thick for students to understand whole, so students pick up some maths classes so they lose the maths understanding.” (31)

“The amount of maths taught at secondary level is too big to enable students to really understand. The result is that they are trying to remember only instead of understand.” (27)

Reason 4. Too optimistic assumptions and expectations of university lecturers (33%)

“... we often expect that (a) all students learn in the same way we did – and that's the best way (b) what students learned at school was the same as we did, and with the same depth.” (51)

Reason 5. Different ways of teaching/learning (30%)

“Students are not prepared to assume responsibility for their learning – rather, they expect continuation of spoon-feeding from high school.” (14)

“Many students have problems adjusting to studying at university. In particular they are used to a teacher planning their study for them and at university they have to do this themselves.” (55)

Reason 6. Lack of mathematics background of mathematics teachers in schools and lack of teaching skills of university lecturers (26%)

“The lack of well qualified teachers of maths in schools means that students do not all get a good background in maths in schools.” (21)

“University staff are appointed because of their knowledge and research record in many cases, not because of their teaching skills, and this puts an extra onus on students to find their own way to understand content.” (38)

Reason 7. Lack of communication between school and university (17%)

“We at Uni level ... aren't keeping up with what is happening at schools.” (48)

“We have almost no communication with the schools. The university and school sectors have almost no overlap.” (51)

“Our ... lack of knowledge and understanding of what is currently taught at school.” (52)

Reason 8. Changes in environment (15%)

“For students, coming to university everything is new: new people, environment, social contexts, norms, expectations. Drastic decline in the amount of personal attention students get from their teachers, compared to high school. Large classes create intimidating situations.” (14)

“Some students cannot cope with the freedom they have being away from home for the first time.” (26)

Question 2. What is your Department doing to reduce the gap?

Table 2. Existing Remedies for Reducing the Gap

Existing remedies for reducing the gap	Percentages
Personal approach	55
Bridging courses	52
Developing different pedagogical strategies	32
Improve communication between school and university	16
Change in assessment: weekly tests, oral exams, detailed feedback	16
Lower the standard	12

Remedy 1. Personal approach (55%)

- Learning support centres
- Small classes
- Individual consultations
- Streaming after diagnostic tests
- Extra tutorials

Remedy 2. Bridging courses (52%)

- Different levels

- Different length
- Different emphasis – e.g. “courses concentrating on mathematical thinking (proof) rather than just pushing content.” (21)

Some participants indicated that the bridging courses often don’t fill the gap. The two main reasons mentioned were:

- The courses are too short in duration. Many bridging courses are just a few weeks or, at most, months long. During that time some students are not able to master the material usually covered at school during several years.
- The mathematical background of the students often is so poor that the emphasis in the bridging courses is on the basics of mathematics: rules, techniques, manipulations, and algorithms. There is no time to teach students higher level of thinking (proofs, reasoning, etc). So the gap in thinking is not filled.

Some universities however offer bridging courses that are much longer, sometimes over 1-2 years that aimed at filling the difference in thinking. In the transition programme described in (Leviatan, 2008) there are 4 units, one semester each:

- Introduction to Advanced Mathematics
- Reading, Writing and Reasoning in Mathematics
- Number Systems
- Definitions and Proofs in Mathematics (the post-calculus stage)

Some of the goals of that programme are:

- An introduction to the mathematical “*culture*”: its language, its logic rules, etc.
- Exposure to typical mathematical activities: generalisations, deductions, definitions, proofs, etc.
- Introducing basic mathematical concepts, such as number systems, sets, functions, sequences, convergence, etc., all these concepts are defined rather vaguely at school.

Remedy 3. Developing different pedagogical strategies (32%)

“We give one lecture on study skills for mathematics and problem solving techniques at the beginning of the year. Each member of our department acts as a mentor to our first year students.” (55)

“Setting weekly homework which is peer assessed in class following lecturer’s working on board, thus trying to encourage an early engagement with new material and a revision of school material.” (17)

“A daily one-hour help session taught by current Masters students but this is basically a patch-up rather than developmental assistance.” (20)

“We aim to take things fairly slowly, with hand-out notes that have detailed explanations. We try to communicate to students just what is expected of them and how to go about achieving their goals in maths.” (51)

Remedy 4. Improve communication between school and university (16%)

“In the summer we offer workshops for teachers, and each year we invite one high school teacher to join our department as a “visiting master teacher” to teach lower level courses and engage with the college faculty in discussions about mathematics and pedagogy. As a consequence, many members of our department explore new methods of teaching; several have received national grants to support this work.” (16)

“In-service day for secondary school teachers. Summer school for year 10, 11 students.” (54)

“Visit high schools and talk about university math courses, and the expectations that those courses place on students ... do sessions for high school students (problem solving sessions and presentations on various math topics).” (14)

“Many of our Department professors go to teach maths at school because they are able to tune proficiently themselves up to the children perception. Professor can imbue the minds of schoolchildren with interest to intriguing, challenging tasks, can give them a taste for non-standard solutions and half-open the curtain to what they will do in the university.” (15)

Remedy 5. Change in assessment: weekly tests, oral exams, detailed feedback (16%)

“We try to give them plenty of feedback on how they're progressing, with assignments and quizzes. We try to listen and respond to as much feedback from students as possible/reasonable in terms of pace, timing of assessment or other administrative areas.” (51)

Remedy 6. Lower the standard (12%)

“Taking out or reducing the mathematics requirements in courses so students can pass!” (38)

Question 3. In your opinion what else can be done to make the transition period smoother?

Table 3. Other ideas to Smooth the Transition

Other ideas to smooth the transition	Percentages
Establish a system to monitor quality at schools and universities	60
Extras: tutorials, courses, learning support, pastoral care, streaming, time (slower pace; ‘adjustment’ semester; summer school)	56
Improve communication between school and university	38
More attention to mathematics education at universities	18

Idea 1. Establish a system to monitor quality at schools and universities (60%)

- Better preparation of school teachers
- Improving school curriculum (less content, more proof, depth and rigour)
- Improving teaching skills of university lecturers - have tertiary teaching qualification

“More deductive (although not necessarily formal) reasoning in high school would help.” “A bit more depth and rigour can be included in school mathematics so that the transition can be smoothened.” (3)

“Making it compulsory for all tertiary teachers of mathematics to have a tertiary teaching qualification as well as their mathematics qualification, hence making them aware of ideas about teaching and strategies for teaching.” (21)

Idea 2. Extras: tutorials, courses, learning support, pastoral care, streaming, time (slower pace; ‘adjustment’ semester; summer school) (56%)

“Involve the university learning centre as much as possible.” (46)

“Talk to first-year university students about how to study *efficiently*. Teach them how to read a maths textbook, i.e. how to do maths on their own.” (14)

“Differentiating among the newcomers, not the first day but after a month or so. Based on the student's own conception of her/his ability, motivation and background and on results from school and the first period at university the student should get an offer to choose among different strands, maybe leading to the same goal but with options for teaching/learning methods, time spent on the material and so on.” (57)

Idea 3. Improve communication between school and university (38%)

“Better contacts with secondary school teachers, in the hope that there could be changes from both sides and especially more information about what is expected from students and how to choose the best preparation for future studies.” (26)

“We must improve the network connecting university mathematics and school mathematics at all levels.” (57)

“Bring final year students into the university to see what it is like here.” (36)

Idea 4. More attention to mathematics education at universities (18%)

- More research in didactics
- Establishing mathematics education units in mathematics departments
- “Set up a Centre with focus on maths/stats education.” (54)
- “Including in mathematics departments a "Mathematics Education Group". Such a group might: legitimize pedagogical studies as a legitimate research area for tertiary teachers of mathematics hold regular educational seminars within the department that others could attend provide support for young faculty members who lack educational expertise mean that some maths ed journals are subscribed to, that would also perhaps influence the culture of the department.” (21)

Conclusions and Limitations

According to participants' responses the major reason for the gap between the school and university education in mathematics is due to the higher level of thinking in university mathematics (72%). The difference is a direct result of where the emphasis is placed by school teachers (calculations and manipulations) and university lecturers (conceptual understanding and rigour). This finding is supported by the theoretical research on the transition issue by Tall (1991, 1997, 2004a, b). As Tall (1991) writes “advanced mathematics, by its very nature, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively. It involves a struggle with inevitable conflicts which require resolution and reconstruction”.

It is clear from the participants' responses that there is a lack of knowledge and awareness by university lecturers of what is happening at school. This shows a need for closer communication between teachers and university lecturers and their institutions, to include understanding of the unique nature of teaching and learning in each sector. To facilitate this will require a mechanism for greater sharing of ideas and practice between the two groups.

As we mentioned earlier the idea of this study has arisen from and is based on teaching practice. It was primarily a practice-based research study with the aim of identifying and promoting pedagogical strategies that may make the transition period smoother and more beneficial for student learning. In their responses, participants presented possible reasons for the gap, and a variety of remedies for bridging the gap that are employed within their own institutions. They have also expressed other ideas on the issue that are worth exploring in the future.

We are well aware about the limitations of the study. We did not do a systematic analysis of the existence of the gap between the school and university mathematics. The study presupposes the existence of the gap based on numerous publications and conference and personal communications that indicated that the gap exists. The sample from the population was taken mainly from lecturers who attend international conferences on mathematics education and therefore have strong interest in and very particular opinion on mathematics education at the university level. For this reason they might not be good representative of the population of university mathematics lecturers. Nevertheless it was interesting to notice that the lecturers from 24 countries with clearly different school environments still express the same concerns. It is hoped that the participants' responses to our questionnaire will cause the readers to reflect on their own teaching practice, and may wish to implement some of the suggested ideas at their own institutions.

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Student Teachers' Understanding of Fundamentals in Mathematics

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In the context of teacher education's emphasis on general pedagogical skills and, in particular, mathematical pedagogical skills, there has been much discussion by mathematics teacher educators on the type and extent of mathematics content knowledge needed by teachers of mathematics for teaching at various school levels. Galbraith, in his 1982 *Educational Studies in Mathematics* paper, conceptualized and studied the *Mathematical Vitality* of prospective teachers from different teacher preparation programmes in Australia. For teacher education programs which include disciplinary mathematics, teacher educators at universities need to question and determine the nature of mathematics content knowledge they are developing in their student teachers. The B.Sc (Ed) program in Singapore is a four-year program which prepares primary as well as secondary school teachers. This paper presents the findings of a study on a small sample of student teachers who have taken disciplinary mathematics as a content subject at university level for five semesters. The instrument used, which is a modification of Galbraith's *Mathematical Vitality* instrument, tests the student teachers' understanding of basic mathematical terminology, of fundamental logic of mathematics and of what constitutes a valid mathematical proof. Of the 20 student teachers, three answered more than half of the 12 items wrongly while eight gave three or fewer wrong answers. The paper will also discuss the responses to individual items, and some implications for the curriculum and teaching of Mathematics to future student teachers.

Keywords: mathematics teachers' subject knowledge, mathematics teacher education, mathematics vitality

Introduction

At university level, mathematics is being taught to an ever-widening spectrum of students who have a slate of diverse reasons and objectives for reading the subject. While few will go on to become research mathematicians, the majority take mathematics courses specially tailored as a service subject in order to complete their degrees in Engineering, Business, Science, Computer Science, Economics and so on. Many of these university undergraduates are thus only interested in Mathematics for its practical applications to their future profession. There is, however, one group of undergraduates for whom mathematics in general will remain relevant and these are student teachers who will be teaching mathematics almost daily in their profession as teachers. In fact, for Singapore mathematics undergraduates of the 1970s and 1980s, teaching was the most natural and popular career upon graduation. While mathematics teachers may not need to understand or use deep level mathematics as is the case of mathematicians, research scientists and engineers, their knowledge of mathematics cannot be at a superficial level of merely carrying out algorithms or applying formulae at a level slightly above what they have to teach in the curriculum. As mathematics

educators, they have the all-important role of developing in their students an early conception of what mathematics is all about. The mathematical preparation of potential mathematics teachers at the university level is thus a significant strand in any discussion of the teaching of university mathematics. As an example, the *Preparation of Primary and Secondary Mathematics Teachers* was one of the working groups in the 1998 ICMI study conference on *the Teaching and Learning of Mathematics at University Level*.

Over the last two to three decades, mathematics teacher educators have wrestled with the issue of content knowledge for teaching with the concept of pedagogical content knowledge (PCK) as specialized knowledge just for teaching which is subject specific and different from subject matter knowledge. There has also been much research and discussion on teacher knowledge and its impact on their teaching and their students' learning (Ball, Lubienski & Mewborn, 2001 and Mewborn, 2003) and reports/reviews on the mathematical preparation of teachers such as Tucker et al (2001). However, there is little conclusive evidence to show links between teachers' mathematics subject knowledge, particularly as measured by the number of mathematics courses taken at university level or highest degree, and their students' achievements.

Ball, Thames & Phelps (2008) have taken the discussion further to distinguish and identify subject matter knowledge (SMK) for teachers, a concept which is not to be confused with pedagogical content knowledge. They were "struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject that is not intertwined with knowledge of pedagogy, students, curriculum or other non-content domains." They looked into specific tasks which teachers are expected to engage in, tasks such as responding to students' "why" questions or recognizing what is involved in representing a particular mathematical idea, and noted that "these tasks require knowing how (mathematical) knowledge is generated and structured in the discipline and how such considerations matter in teaching", at the same time lamenting that the knowledge and skills for such tasks are "not typically taught to teachers in the course of their formal mathematics preparation".

The concept of *Mathematical Vitality* as an attribute of a mathematically aware student of mathematics was defined by Galbraith (1982) and this attribute included having a deeper understanding of mathematical processes, the ability to carry out a mathematical analysis and the ability to construct a logical defence of mathematical statements. These are expected abilities of university graduates who major in mathematics. While students can graduate with a university mathematics degree with the knowledge of much mathematics content, what lasts much longer is this *Mathematical Vitality* because it is an attribute which becomes part of the learner's mathematical disposition.

We thus conceptualized *Mathematical Vitality* as the attribute of having basic understanding of the discipline of mathematics in terms of its language and structure, and in particular, the ways and processes of determining and accepting mathematical truths. As there has been research evidence to show that proofs are difficult, not only for potential mathematics teachers but also for university mathematics undergraduates in general (Jones, 2000, Knuth, 2002 and Weber, 2003), we also include an understanding of mathematical proofs as a component of *Mathematical Vitality*. However, it should be pointed out that at this stage, our study of *Mathematical Vitality* is not focused on student teachers' conceptions of proof or their particular difficulties with proofs as discussed in Knuth (2002) or Weber (2003), but on their basic understanding of when a proof is considered valid mathematically.

Although *Mathematical Vitality* may not be content in terms of mathematical topics,

theorems and definitions, it is certainly a constituent part of subject matter knowledge for teachers as described by Ball, Thames and Phelps (2008). This is because *Mathematical Vitality* does not deal with knowledge of pedagogy or student understanding, but is important for teachers to have in order that they can in turn help their students build up some understanding of the structures of the discipline.

As mathematics teacher educators at Singapore's National Institute of Education (NIE), the sole teacher education institute in Singapore responsible for pre-service teacher preparation, we are particularly concerned with the mathematical preparation of future mathematics teachers. In particular, for student teachers who read disciplinary mathematics in their program, there was concern as to whether they would graduate with the understanding of these fundamentals and whether their teacher preparation program had adequately developed *Mathematical Vitality* in these potential mathematics teachers. In this context, we undertook a study *Mathematical Vitality of Mathematics Student Teachers* to determine the state of our student teachers' *Mathematical Vitality* at various points of their tertiary mathematics journey. This paper reports on the initial findings on the senior student teachers' *Mathematical Vitality*. Subsequent phases of the study will focus on intervention in terms of curriculum and teaching.

Context of the Study

The NIE has three groups of programs that prepare teachers for teaching various subjects in the school system. This study is concerned with the four-year Bachelor of Science (Education) ((B.Sc (Ed)) program where there is adequate curriculum space and time for the development of student teachers' mathematics content knowledge. The program has two tracks, for primary teaching and secondary teaching respectively.

All programs for teacher preparation at NIE include two or three Curriculum Studies (CS) subjects, each of which comprise methods courses for developing student teachers' pedagogical content knowledge and pedagogical skills necessary for teaching that specific subject. The number of CS subjects taken by the B.Sc (Ed) student teachers is two for potential secondary teachers and three for potential primary teachers. In addition, the undergraduates in the B.Sc (Ed) program have to take Academic Subjects (AS) which are normal university level disciplinary subjects. Those in the secondary track take two such subjects, one at major and the other at minor level, and these two subjects correspond to their CS subjects, for example, Mathematics and Biology. Those in the primary track take one AS which is usually but not necessarily linked to one of the three teaching subjects. They are also required to take three subject knowledge (SK) areas which correspond to the three teaching subjects. The courses in the SK areas are different from normal university level discipline courses. They cover content which is related to primary school curriculum but with a deeper treatment and from a broader contextual perspective. Subject knowledge courses are not deemed necessary for the secondary track since the academic subjects take care of SMK for the student teachers. Table 1 shows program structure of the four-year B.Sc (Ed) program. A subject comprises a few courses and courses are normally of two or three academic units (AUs) where 1 AU is equivalent to 12 hours of contact teaching time.

The structure shows that the academic subject forms a large portion of the program, 30% when taken at major level. So, a primary track potential mathematics teacher who also takes AS mathematics would spend 55 AUs or 43% of his/her program on mathematics courses: 39 AUs for AS mathematics, 10 AUs for CS mathematics and 6

AUs for SK mathematics⁸. In the secondary track, a potential mathematics teacher has to take both AS and CS mathematics although the AS mathematics could be at either major or minor level.

Table 1. Structure of the B.Sc(Ed) Program

Track	Subject Specific			Compulsory Courses			TOTAL
	CS	AS*	SK	Education Studies	Practicum	Others	
Primary	3 subjects (10 AUs each)	1 subject (39 AUs)	3 subjects (6 AUs each)	5 courses (12 AUs)	4 stints (21 AUs)	3 courses (8 AUs)	128 AUs
Secondary	2 subjects (12 AUs each)	2 subjects Major: 39 AUs Minor: 24 AUs	Nil	5 courses (12 AUs)	4 stints (21 AUs)	4 courses (11 AUs)	131 AUs

Note: Each AS subject comprises 13 courses of 3 AUs each. A subset of 8 courses is taken by those doing it at minor level.

Within the B.Sc (Ed) program, all mathematics courses are offered by the Mathematics and Mathematics Education (MME) academic group, the department which houses both mathematicians and mathematics educators and is responsible for the curriculum and delivery of these courses. MME has the main responsibility for the development of mathematical SMK and PCK of the B.Sc (Ed) student teachers and hence it is important for the faculty to understand the mathematical standards of our student teachers beyond assessment results and to investigate if our graduates do indeed achieve the stated objectives of our courses. Moreover, since our undergraduates are being prepared for just the teaching career, MME can design and deliver focused AS mathematics courses which are better suited for teachers, provided it is clear on what is required to develop good teachers of mathematics.

Method, Instrument and Sample

In this context, the current study is designed as the first stage in finding out the *Mathematical Vitality* of our student teachers where we have been responsible for their mathematics learning at tertiary level. The authors sought and received permission to modify and use items from the *Mathematical Vitality Test* as designed by Galbraith (1982). The instrument comprised 12 items in multiple-choice format and space was provided for explanations if any. The items are given in the Annex together with an explanation of what mathematical understanding was being examined for each item. The instrument also sought background data of the participants in terms of the mathematics courses which they had completed in their program so far.

The population of this study is the whole cohort of student teachers of the B.Sc(Ed) program who take AS mathematics as a major. They were from both the primary and secondary tracks and the study was conducted when they had completed at least ten

⁸ It must be noted that while any AS mathematics student teacher in the primary track would always choose CS mathematics and correspondingly SK mathematics, the converse is not true since a primary track student teacher is trained to teach 3 subjects i.e. has 3 CS and 3 SK subjects, but has only 1 academic subject which could be any one from the slate of Mathematics, Physics, Chemistry, Biology, English, English Literature, History, Geography, Art, Music or Drama.

mathematics courses out of the thirteen required for AS mathematics. They were invited to participate through taking the test on a voluntary basis and were assured that the results would have no bearing on their course assessment and moreover produce findings to help MME improve the mathematics curriculum. The test could be taken at any time during a two-day period at a fixed venue. Since the student teachers knew that it was for research purposes, there was no concern that earlier participants would tell others about the items and this flexibility in test administration time was intended to encourage higher rates of participation. Nevertheless, of the more than eighty student teachers in the cohort, only twenty turned up to take part in the study.

Findings and Discussion of Items

General Results

Among the 20 participants, nine were from the primary track and eleven were from the secondary track. Fourteen of them have taken ten mathematics courses while the remaining six had completed eleven or twelve courses. Eight of these courses were compulsory core courses (Calculus 1 and 2, Algebra 1 and 2, Statistics 1, Finite Mathematics, Number Theory and Computational Mathematics) while the other courses were electives with different combinations of choices for the various student teachers.

Although there was no time restriction, most of the student teachers completed the test in 20 to 30 minutes with three taking only 10 to 15 minutes and two taking 45 minutes. The results showed a wide variation among the student teachers. The test instrument had 12 items and the number of correctly answered items ranged from 2 to 11. The items are not given different marks based on difficulty and here the score on the test shall be taken to mean the number of correctly answered items.

Figure 1 shows the number of student teachers attaining each score from 0 to 12. The mean score is 7.7, the median is 7.5 and the mode is 7 where 5 student teachers obtained correct answers to 7 items. Half the student teachers managed to get 8 or more items correct while only 3 of them got less than 6 correct. There was one outlier case who obtained only two correct answers.

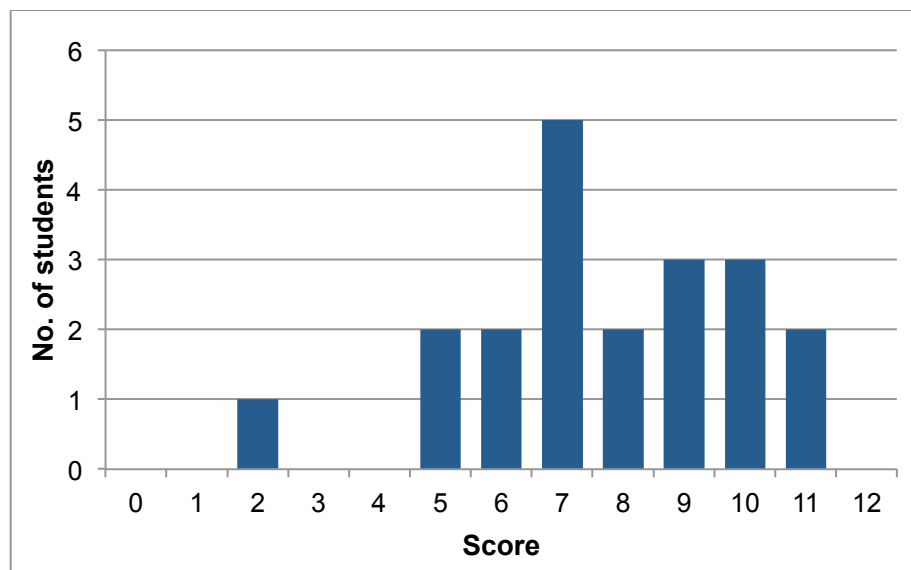


Figure 1. Distribution of student teachers across the scores.

Discussion of individual items

Table 2 shows the performance of student teachers on individual items with respect to the choices made. The correct choice is marked in bold and underlined. From table 2, it can be seen that 6 of the items, items 1, 4, 6, 7, 10 and 11, had at least 70% choosing the correct answer. In fact, items 1 and 11 were the items where almost everyone selected the correct option and the only person who chose a wrong option in item 1 was the student teacher who only obtained two correct answers among the 12 items.

Table 2. Student teachers' choice of options in the items

Item Number	Choice				
	A	B	C	D	E
1	<u>19</u>	0	0	1	0
2	13	<u>7</u>	0	0	0
3	3	3	3	5	<u>6</u>
4	2	0	0	<u>14</u>	4
5	<u>11</u>	6	0	2	1
6	0	2	1	2	<u>15</u>
7	<u>14</u>	1	1	3	1
8	0	<u>12</u>	5	0	3
9	5	1	<u>10</u>	4	
10	<u>16</u>	1	2	1	
11	0	0	0	<u>20</u>	0
12	2	<u>10</u>	1	7	

The following sections will discuss performance in the various items clustered according to the concepts or processes being tested.

Items testing basic knowledge

There were two items designed to test basic knowledge. These were item 2 which tested the knowledge of convention that the square root sign ($\sqrt{}$) referred to the positive root only and item 4 which checked on their knowledge that π is irrational and the consequences thereof. The students performed fairly well on item 4 although it is of concern that 6 of them either did not realise that π is irrational or could not deduce that the product of a rational number and an irrational number is necessarily irrational.

Item 2 was the single item where the wrong option (A) was chosen by a sizeable majority. In their earlier learning of mathematics, they had been so conditioned to write down both positive and negative square roots when solving $x^2 = a$ (where $a > 0$) that it was quite natural for 13 of them to think that $\sqrt{9}$ was ± 3 . This result is not surprising as similar results were reported for the same item in Galbraith (1982) while testing prospective teachers in Australia and in Wong (1990) for testing PGDE (Sec) student teachers in Singapore. However, this is not too worrying since the misconception can be corrected through explicitly making the notation clear and emphasising the distinction between $y^2 = x$, $y = \pm \sqrt{x}$, $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Items testing the use of examples as proof

Items 3, 11 and 12 tested whether the participants would accept a range of examples

(whether a finite or infinite number of them) as proof of a result. For item 3, it is clear from the spread of choices and that only 6 chose the correct option (E) that this was a difficult item for the student teachers. It is also an item where every distractor was chosen by a few student teachers. Distractors (A), (B) and (C) were designed to test the misconception that an infinite number of objects satisfying a property points towards all such objects satisfying the property and it is a matter of concern that a total of 9 participants chose these options. Very few wrote additional explanations but it was worrying that one wrote “infinitely many means all”. There were 5 student teachers who chose option D and a closer study of the four explanations given showed that they did not have the same misconception. In fact two wrote “infinitely many does not imply all” while two others tried to give counter-examples such as right-angled triangle satisfying Pythagoras’ Theorem. However, a lack of understanding of the phrase “necessarily correct” and not taking into consideration the phrase “more likely” resulted in their choosing Option (D) as the closest to their conclusion.

In item 12, however, a classroom situation and actual well-known geometrical result were presented and the participants were to determine if the teaching activity was an acceptable “proof” of the result. Half of the 20 participants stated correctly that although many examples were used to verify the result, the proof was invalid as one must prove it for ALL triangles. Nevertheless, 7 of the participants did accept the activity as valid because it covered different types of triangles but it was reassuring that 3 of the 7 explained that they felt the activity was valid in the context of teaching but more formal proofs were necessary for mathematics.

While student teachers had problems with item 3 and, to a lesser extent item 12, they had no problems with item 11. Perhaps this is due to the fact that the “proofs” used specific numerical examples while items 3 and 12 did not and hence all the participants were able to determine that the “proofs” were invalid.

Items testing mathematical logic

Items 5 to 8 checked on the participants’ ability to apply mathematical logic to particular examples. In general, items 6 and 7 were well done. For item 6, 15 of the participants understanding that a proven result does not imply the truth or otherwise if its converse and for item 7, 14 of the participants understood how to select a counterexample to disprove an if-then statement.

For items 5 and 8, 11 and 12 participants respectively obtained correct answers, with 8 participants getting both correct and 5 getting both wrong. What was particularly interesting was that among the 5 participants who selected Option (C) in item 8, believing that the converse of a true statement was also true, 4 of them also gave the wrong option in item 5 with three of the four accepting that the contrapositive of a theorem was true but felt it still needed to be proved.

Items dealing with specific proofs

Items 9, 10 and 11 dealt with specific proofs of results and student teachers were to determine whether the proofs were correct or valid and where there were errors. Item 11 has already been mentioned and Item 10 was also not difficult, with 16 student teachers being able to ascertain that the proof by contradiction was valid. In fact, 5 of the 16 wrote in explanation that the given example was a *proof by contradiction*.

However, only 10 participants managed to recognise that the proof in item 9 was incorrect because it was proving the converse instead of the result, choosing option (C).

A closer inspection of the answer scripts showed that 3 of these 10 also wrote explanations which contained some mathematical error.

Of the remaining 10 who did not choose option (C), 5 deemed the proof valid (option (A)) while 4 chose (D) but either gave no reason or mathematically incorrect reasons.

Conclusion and Implications

Overall, the results showed that student teachers were not very strong in their understanding of fundamental mathematical logic, mathematical language or proof techniques. Considering that the participants had already completed at least ten of their thirteen courses for the academic subject mathematics, only half of them managed to get 8 or more items correct out of the 12 items. The difficulty could be due to two features of the instrument: (i) many of the questions were not contextualised with explicit numerical examples or specific geometrical properties and (ii) the rather unfamiliar language used. This is also a cause for concern because it meant that student teachers lacked fluency in mathematical language and were unable to draw general conclusions.

In terms of the concepts tested, the better performance of item 6 relative to item 9 indicated that while they understood that one could not make conclusions about the converse of a proven statement, a substantial number could not determine that the given proof was proving the converse of a statement instead of the statement itself and was thus a wrong proof. In the experience of faculty teaching university mathematics, this result is probably not surprising as students are quite likely to construct a proof from the conclusion to the premise instead of the other way round. This could be due to the fact that most of the theorems learned in secondary or high school were the if-and-only-if type of results.

It also appears that mathematical logic is not well understood by the student teachers as seen by their inability to argue out the correct responses, especially where the examples were not specific enough. The relationship between a statement and its contrapositive, for example, as tested in items 5 and 8, was only understood by slightly more than half the participants. It has been a debated issue in the department whether various aspects of mathematical logic coupled with methods of proofs should be covered in a specific separate course or whether they should be just taught within the various courses covering different topics. While it would be difficult to teach the logic without content topics, the current practice of having them implicitly done within various topics has not been successful as shown by these results. Faculty could make a concerted effort to continually emphasise the various methods of proof and the understanding of in mathematical logic in their teaching of topics, to the extent of explicitly naming the methods or terms. From the fact that some students actually used terms like “contrapositive” and “proof by contradiction”, we deduce that some of their mathematics lecturers had taught the terms but as indicated by those who could not select the right options, the understanding of fundamental mathematical ideas and processes of proof has not been internalised by all those who have taken a substantial number of mathematics courses.

The findings could not be directly compared to the studies by Galbraith (1982) and Wong (1990) but for some similar items, a greater proportion of the students in the current study were able to select the correct answer as compared to those in Wong (1990). This could be due to the currency of their tertiary mathematics courses, because the sample in Wong’s study comprised students who had graduated one or more years previously, while the students in the current sample are still enrolled in undergraduate mathematics courses. Galbraith (1982) has argued that pre-service teachers seemed to

approach mathematics with an instrumental view and has also conjectured that simply taking more mathematics courses would not enhance *Mathematical Vitality*. He had suggested more research into establishing a link between student performance and *Mathematical Vitality* as a measure of relational mathematics, both of the teachers themselves and their approach to teaching mathematics. Further study would need to be taken after the students in this study graduate to (a) know if they have retained their *Mathematical Vitality* and (b) establish links, if any, between their *Mathematical Vitality* and their teaching approaches.

One specific limitation of this set of data is the small sample size and the researchers will work towards persuading more student teachers to participate in the next data collection which will be among the new first-year students. Nevertheless, although the findings cannot be generalised, they are useful for informing faculty of their students' mathematical understanding.

As is the case with most tertiary institutions, mathematics courses are taught by experts in each area and it is quite the norm for each faculty member to cover a systematic collection of definitions, theorems and applications with the assumption that their students will somehow, through doing mathematics, learn to be fluent in mathematical language and develop a general understanding of the logical processes of arriving at mathematical truths. While this may be the case, there are students who fail to acquire such *Mathematical Vitality*, as is shown through the findings of this study, although they had performed better than those in an earlier study. The authors conjecture that enhancing the *Mathematical Vitality* of undergraduates is possible but requires a concerted, conscious and consistent endeavour on the part of faculty members in at least a large majority of mathematics courses to do so through agreed means and approaches. The results of the current study form the starting point for just such an endeavour, and the fact that mathematicians and mathematics educators work together in the same department holds promise of successful collaboration.

Continual curriculum review and investigation into our own teaching and our student teachers' learning are necessary for mathematics teacher education to remain relevant and effective. In addition to considering international issues, for example those documented by working groups 1 and 2 of the ICMI Centenary Symposium (Menghini et al, 2008), local findings on mathematical understanding of our student teachers will contribute towards targeted and specifically designed enhancements which will not be based on fashion or political correctness but on professional evidence-based and informed judgement. Further intervention study on modified curriculum or teaching methods will be the next phase of this research study.

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Annex A Mathematical Vitality Test Items

This test comprises 12 multiple choice questions. For each question, select only **one answer** (A, B, C, D or E). You are encouraged to explain your answer in the space provided.

Item No	Question	What is being tested
1	Which of the following statements is correct? (A) If $x = 4$, then $x^2 = 16$. (B) If $x^2 = 16$, then $x = 4$. (C) $x = 4$ if and only if $x^2 = 16$. (D) If $x \neq 4$, then $x^2 \neq 16$. (E) None of the above.	Understanding of the if-then logic
2	Which of the following statements is correct? (A) $\sqrt{9} = \pm 3$ (B) $\sqrt{9} = +3$ (C) $\sqrt{9} = -3$ (D) $\sqrt{9} = 9 $ (E) None of the above	Knowledge of basic symbol/terminology of square root sign
3	It has been proven that infinitely many triangles possess property H. <i>Statement S</i> : All triangles possess property H. Which of the following statements is <i>necessarily</i> correct? (A) <i>Statement S</i> is true but further proof is needed. (B) <i>Statement S</i> is true and no further proof is needed. (C) <i>Statement S</i> is more likely to be true than false. (D) <i>Statement S</i> is more likely to be false than true. (E) None of the above.	Having an infinite number of the object satisfying the statement has no bearing on the truth of the statement applied to all such objects.
4	The radius of a circle is k cm where k is a positive integer. Select whichever of the following you believe to be correct. (A) The circumference could be exactly 88 cm. (B) The circumference could be exactly 9 cm. (C) Either (A) or (B) could be correct. (D) Neither (A) nor (B) could be correct. (E) It is not possible to decide in favour of any of the above alternatives.	Knowledge that π is irrational and that the product of a rational with an irrational must be irrational.
5	<i>Theorem X</i> : If three positive integers a , b and c satisfy condition P, then they satisfy condition Q. <i>Statement Y</i> : If three positive integers a , b and c do not satisfy condition Q, then they do not satisfy condition P. <i>Theorem X</i> has been proved. Which of the following statements is <i>necessarily</i> correct concerning <i>Statement Y</i> ? (A) <i>Statement Y</i> is true and does not need further proof. (B) <i>Statement Y</i> is true but needs to be proved. (C) <i>Statement Y</i> is false and does not need disproof. (D) <i>Statement Y</i> is false but need disproof. (E) None of the above.	Understanding the relation of a statement's contrapositive to the original statement.

- 6 The following are two statements:
Statement X: If a polygon has property P, then it has property Q.
Statement Y: If a polygon has property Q, then it has property P.
Statement X is a theorem in geometry which has been proved. Which of the following statements is *necessarily* correct concerning *Statement Y*?
 (A) *Statement Y* is true and does not need further proof
 (B) *Statement Y* is true but needs to be proved.
 (C) *Statement Y* is false and does not need disproof.
 (D) *Statement Y* is false but needs disproof.
 (E) It is not possible to determine whether *Statement Y* is true or false from the information given.
7. A statement *S* reads as follows:
S: A whole number *n* is divisible by 6 if the sum of its digits is divisible by 6.
 Select whichever of the following you believe to be true.
 (A) The number 39 shows that *S* is false.
 (B) The number 42 shows that *S* is false.
 (C) The numbers 39 and 42 both show that *S* is false.
 (D) *S* is false but neither 39 nor 42 is adequate to disprove it.
 (E) *S* is true.
- 8 We know that a given statement *S* is true if $y < 0$.
 Which of the following statements must be true?
 (A) If *S* is true, then $y \geq 0$.
 (B) If *S* is false, then $y \geq 0$.
 (C) If *S* is true, then $y < 0$.
 (D) If *S* is false, then $y < 0$.
 (E) None of the above.
- 9 Result: For $a > 0$ and $b > 0$, $\frac{1}{2}(a + b) \geq \sqrt{ab}$.
Proof: Step 1: If $\frac{1}{2}(a + b) \geq \sqrt{ab}$, then $(a + b)^2 \geq 4ab$.
 Step 2: Hence, $a^2 - 2ab + b^2 \geq 0$.
 i.e. $(a - b)^2 \geq 0$.
 Step 3: Since $(a - b)^2 \geq 0$ is always true,
 $\frac{1}{2}(a + b) \geq \sqrt{ab}$.
 Do you think the proof is valid? Why?
 (A) Yes, the proof is valid.
 (B) No, Step 1 is wrong because we cannot square both sides of an inequality.
 (C) No, the proof should not begin with what is to be proved and lead to a correct result.
 (D) No, for other reasons.
 (Please give/explain your reason(s).)

Understanding of the truth of a statement has no bearing on that of its converse.

Ability to use counter examples appropriately to disprove an if-then statement.

Ability to apply the relationship between a statement and its contrapositive.

Ability to recognise the error of proving the converse instead.

- 10 Result: If $x + 4$ is an odd integer, then x is an odd integer.
Proof:
Given that $x + 4$ is an odd integer, assume that x is even.
Let $x = 2k$ where k is an integer.
Then $x + 4 = 2k + 4 = 2(k + 2)$ which is even, since $(k + 2)$ is an integer.
Since this contradicts the fact that $x + 4$ is odd, the assumption is wrong and hence x is odd.
Which of the following statements is correct?
(A) The proof is valid.
(B) The proof is wrong because it begins with a wrong assumption.
(C) The proof is wrong because we are actually proving that if x is even, then $x + 4$ is even.
(D) The proof is wrong but for other reasons. (Please give/explain your reason(s).)
- 11 Result: If m is a factor of n and n is a factor of k , then m is a factor of k .
Proof 1:
Consider the numbers: 4, 8 and 24.
4 is a factor of 8 since $4 \times 2 = 8$.
8 is a factor of 24 since $8 \times 3 = 24$.
Hence $24 = 4 \times 2 \times 3$. So 4 is a factor of 24.
Proof 2:
3 is a factor of 6 and 6 is a factor of 24. We see that 3 is a factor of 24.
3 is not a factor of 5 and 5 is not a factor of 7. We see that 3 is not a factor of 7.
Which of the following statements is correct?
(A) Proof 1 only shows one case. Proof 2 is valid because it shows both factors and non factors.
(B) Proof 2 is wrong because the result is not about non-factors. Proof 1 is correct because it deals with numbers which meet the conditions.
(C) Both proofs are valid, they just use different numbers.
(D) Both proofs are not valid because we need to show that the result is true for all numbers m , n and k .
(E) None of the above.
- Recognising a proof by contradiction and its validity.
- Recognising that numerical examples are not proofs.

12. Result: The sum of the measures of the interior angles of a triangle is 180° .

Proof:

Forty pupils of a class each drew a triangle. Some drew acute-angled triangles, some drew obtuse triangles and others drew right-angled triangles. They cut out their triangles, tore them up and put the three angles together as in the diagram below.



Since the three angles lay on a straight line, they add up to 180° .

Is the above a valid proof for the result?

- (A) No, diagrams cannot be used in proofs.
 - (B) No, because only 40 triangles were used and we must prove for all triangles.
 - (C) No, because there are no mathematical statements in the proof.
 - (D) Yes, because different types of triangles were used to verify the result.
-

Recognising that a good range of examples does not constitute a valid mathematical proof.

Student Understanding of Riemann Integration: The Role of the Dynamic Software GeoGebra

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Issues surrounding the transition from school to university study in mathematics have been brought into sharper focus in recent years. One aspect of this is that schools may not teach certain useful topics, or they may not teach them to the depth universities would prefer. One such topic, considered here, is that of Riemann integration and the Fundamental Theorem of Calculus. We discuss a controlled experiment in which a module of work based on the program GeoGebra was used with a group of secondary Year 13 (age 17 years) students to assist them with understanding the concept of Riemann integration. The results show that while their understanding improved significantly overall, and especially on graphical questions, this was not true for related procedural or algorithmic skills. Possible reasons for these outcomes are presented.

Keywords: Riemann integration; GeoGebra; calculus; transition, versatile thinking; concept image

Background

Vinner and Hershkowitz [1] first introduced the notion of concept image in mathematics, defining it as the total cognitive structure associated with the concept, including all the mental pictures and associated properties and processes. Thus it is a dynamic entity that develops differentially over students, through a multitude of experiences [2]. In connection with the notion of concept image, Tall [3] has used the term “met-before” to describe a mental facility based on specific prior experiences of the individual. These met-befores influence an individual’s interpretations of new situations and thus their personal growth of mathematical thinking, sometimes advantageously, but sometimes producing conflict, which Vinner [4] suggests is necessary to advance students to a higher intellectual stage

Students’ concept image of a definite integral may consist of geometric definition and analytic ideas, with the former including the signed area of a region under a graph and area-so-far, whereas the latter involves limits of Riemann sums. However, some studies [5, 6] indicate that the intuition inherent in concept images dominates learning. Often students appear to have a procedural understanding of the definite integral, with the notion of anti-derivative as the key idea. They seem to respond better when asked to give a geometric definition of the integral [6], although it is often associated with a canonical image of a function situated above the x -axis [6]. Students may understand the definite integral to be the area under the curve, but seem unable to connect this area to the numerical sequence of Riemann sums [6].

Hence, in order to help students to develop a rich and comprehensive concept image of a definite integral, coordination between the visual schema of Riemann sums and the analytical schema of the limit of the numerical sequence could prove helpful [7]. It has also been proposed that it is important to develop an embodied cognition approach to the notion of infinity and limit [8], doing so through carefully chosen

examples using modulus function, the integer-value function and piecewise functions, since such cases extend students' previous experience and broaden their concept image of definite integral [5]. Hence this study sought to help students build a richer concept image of Riemann integral by combining visual and analytic activities.

Theoretical Framework

Since fluency of representational translation or conversion [9], lies at the heart of what it means to 'understand' many of the more important underlying mathematical constructs, a priority of mathematics education should be to increase the ability of students' to formulate multiple representations. However, delineating what should be involved in encouraging fluency with representations is not so clear. In his framework, which he calls *representational versatility* [10, 11], Thomas includes: addressing the links between representations of the same concept; the need for both conceptual and procedural interactions with any given representation; and the power of visualization in the use of representations.

Tall has developed ideas about encouraging the inclusion of embodied approaches to mathematics alongside symbolic and formal ones [12], into a framework of the cognitive development of mathematical thinking [3, 13] that he calls the Three Worlds of Thinking. In this framework he elucidates thinking and learning as taking place in three worlds: the embodied; the symbolic; and the formal. The embodied world relates to Bruner's [14] iconic and enactive modes of representation of knowledge, and involves making use of visual and physical attributes of concepts, combined with enactive sensual experiences to build mental conceptions. The symbolic world is where symbolic representations of concepts are acted upon, or manipulated. There is a change from doing mathematics through processes to thinking about mathematical concepts [13] that is facilitated by powerful symbolic forms, called *procepts* [15], such as $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$. These may be thought of as processes or concepts, with the duality bridging the two qualitatively different modes of thought. In this way moving from the embodied to the symbolic world changes the focus of learning from the physical to properties of symbols and their relationships. In the formal world properties of objects become axioms, and learning comprises the building and proving theorems by logical deduction.

Since calculus, including Riemann integral, historically has geometric origins relating to area, symbolic manipulation of sums, as well as formal definition and proof, the three worlds framework was considered ideal to analyse student thinking and learning about it. The research questions in this study were 'What difference, if any, does use of computer software such as GeoGebra in teaching make to the learning of the integration?' and 'Will such an approach improve conceptual understanding of integration?'

Method

This study comprised a mixed method design [16], with quantitative data taken from a pre-test, post-test control group experiment (pre-test and post-test results) combined with qualitative data from case studies, comprising interviews.

Participants

The 42 participants came from two Year 13 (age 17 years) calculus classes in their last year of high school at a high socio-economic level all-girls school in Auckland,

New Zealand. Two mixed-ability classes were chosen as experimental and control groups, the former consisting of 23 students who were taught the topic of definite integral by the first-named researcher. The control group comprised 19 students and these learned the topic of definite integral with their usual teacher, who adopted a traditional method of teaching. Both groups were ethnically diverse, with approximately 60% Asian (predominately Chinese and Korean), 30% New Zealand or other European and 10% other (including Indian, Filipino and Maori) students.

The students in the control group were mostly average-achieving students who displayed a satisfactory level of understanding in Year 12 (age 16 years) mathematics and were progressing well in Year 13 calculus course. In contrast, there were some students in the experimental group who were struggling to cope with the content taught in the Year 13 calculus. Evidence for this can be seen in their previous year's results where 83% of the students in the control group gained more than 20 credits in Level 2 Mathematics (Year 12 external assessment), whereas only 73% of the students in the experimental obtained more than 20 credits. In addition, three experimental group students gained fewer than 15 mathematics credits and five did not achieve either the Algebra or Calculus Achievement Standards. In contrast, for the control group, only one student had fewer than 15 credits, and two did not achieve either standard.

Instruments and approaches

The technology used in the study was the open-source mathematical software, GeoGebra, which combines the features of dynamic geometry software (DGS) and a computer algebra system (CAS) in a single integrated program [17] offering algebra, graphics and spreadsheet representations in a fully connected environment (see Figure 1).

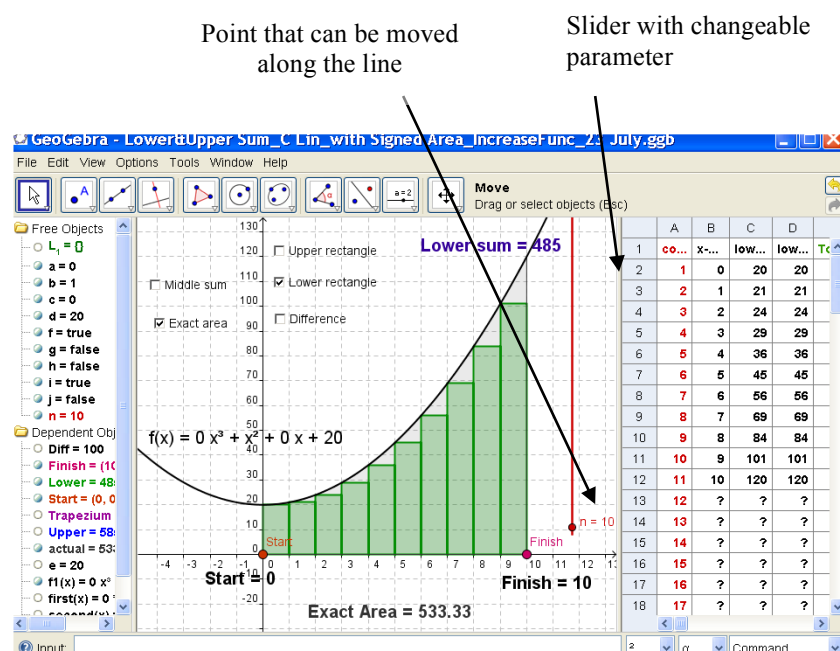


Figure 1. GeoGebra interface with three representations – algebra, graphics and spreadsheet views.

The teaching-learning sequence implemented with the experimental group

consisted of five 40-50 minute lessons on the topic of integration. The main features included were: distance covered as a special case of the integral; the numerical accumulation process (limit of Riemann sum); area under a curve; the relation between a graph of a function and its integral (properties of integrals); the “area-so-far” function; and the relation between integration and anti-derivative. The aim of the instruction was that students build a rich concept image for integration and develop a versatile understanding of integration. The participants in the control group were taught the same topic by their usual teacher within the same amount of time. She taught the class with her standard approach, which involved some technology, such as an Interactive White Board, but she agreed not to use GeoGebra or similar software (e.g. Autograph).

The researchers constructed GeoGebra files that allow one to visualise dynamically the concept of Riemann integral using lower and upper sums, and explore different aspects of the Riemann integral. The objectives of the student activities were to: (1) learn how integrals relate to Riemann sums; (2) learn to use numerical methods to calculate approximations of a definite integral; (3) help students understand the notion of the “signed area” as a limit of a Riemann Sum; (4) allow them to explore properties of the definite integral; and (5) gain understanding of the ‘area-so-far function’ by tracing the cumulative area (see Figure 2).

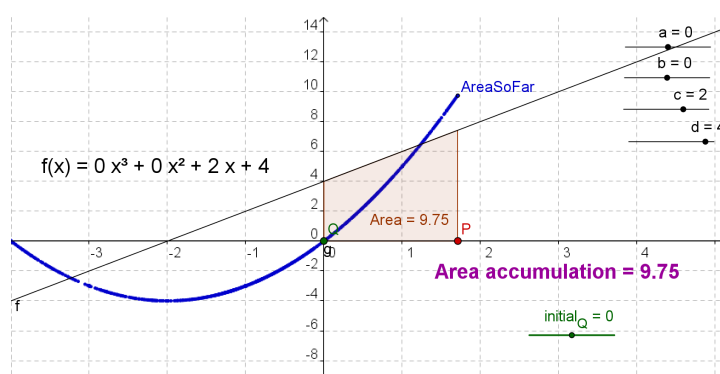


Figure 2: The GeoGebra program as used for tracing the area-so-far functions.

The Pre-test and the post-test

Two written tests were constructed and used as pre- and post-test measures. The pre-test contained eight questions worth a total of 15 marks relating to several different areas. Some questions were designed to measure students’ visual understanding of definite integral, while others were constructed to explore students’ concept image of integral and to see if they were able to use the integral to calculate simple areas. Further, questions were used to find out whether the students could rely on visualisation of the graph of the function to handle the calculation of an integral. Lastly, two questions in the pre-test were typical procedural Year 12 Integration problems and were constructed simply to see whether the students were, or were not, able to evaluate and calculate an indefinite and definite integral.

The post-test was designed to compare the improvement of the students’ understanding of the notion of integral and Table 1 outlines its coverage. It was constructed in a similar manner to the pre-test but with concepts related to definite integral and the Fundamental Theorem of Calculus (see Figure 3 for examples). It also contained more questions than the pre-test since the aim was to probe the students’

understanding of integrals in more detail, including whether the students understood various representational forms of integrals and whether they were able to translate from one representation to another. In addition, the different approaches the students used to solve the problems in the post-test were analysed in order to compare the learning approach between the experimental group and the control group.

Table 1: The Structure of the Questions Used in the Post-test

Section	Questions	Content tested	Total marks
A	1, 2 and 3	Properties of definite integral	9
B	4(i), 5, 6(i) and 6(ii)	Understanding of Riemann sums and the use of numerical methods	10
C	7, 8, 10, 12(i) and 12(ii)	Understanding of the “signed area”	10
D	9(i) and 9(ii)	Understanding of the “area-so-far” functions	4
E	11	Understanding of the Fundamental Theorem of Calculus	2
F	4(ii)	Symbolic manipulation of definite integral	2
G	4(iii), 12(iii) and 13	Use of multiple representations (representational versatility)	8

The interviews

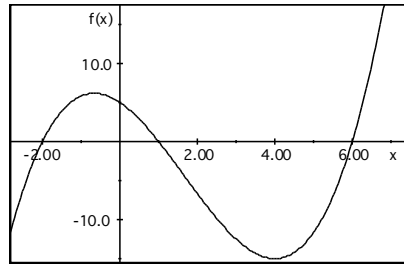
After both groups had completed the post-test, six participants from the experimental group representative of high (2), average (1), low (2) and negative (1) improvement were selected for a semi-structured interview. During the 25-minute interviews, students were asked for their opinions on the teaching approach and whether the use of GeoGebra had influenced their understanding of integration. Each interviewee was also asked questions based on her answers in the post-test, where they were either erroneous or revealed interesting views needing explanation. They were also questioned about whether their perspectives and understanding of the concept of integral had changed.

- If $\int_0^4 f(x)dx = 20$, then write down the value of:
(i) $\int_1^5 f(x-1)dx$ (ii) $\int_0^4 (f(x) + 3)dx$ (iii) $\int_4^0 (f(x) - 2)dx$
- If $\int_a^b g(x)dx = 10$ and the graph of $g(x)$ is translated vertically by 5 units upward to give the graph of $h(x)$. Write down the value of $\int_a^b h(x)dx$ in terms of a and b .
- i) The table below shows the value for the function $f(x) = x^2 - 2$ for $x = 0, 2, 4, 6, 8$ and 10 . Find an approximation to the integral $\int_0^{10} (x^2 - 2)dx$ using a Riemann sum with left-endpoints and 5 equal sub-intervals (i.e. of 5 rectangles under the curve with equal width).

x	0	2	4	6	8	10
$x^2 - 2$	-2	2	14	34	62	98

- ii) Evaluate $\int_0^{10} (x^2 - 2)dx$.
- If $f(x)$ is a strictly increasing function (i.e. $f(b) > f(a)$ if $b > a$), an approximation to $\int_1^4 f(x)dx$ using a Riemann sum (i.e. of rectangles under the curve) with left-endpoints and number of rectangles $n = 10$ works out to be 7.62. Will an approximation with left-endpoints and number of rectangles $n = 50$ be: (a) more than 7.62 (b) less than 7.62 (c) equal to 7.62 (d) not possible to say
Explain the reason for your answer.

8.



The graph of $y = f(x)$ is shown as above. If $\int_{-1}^a f(x)dx = 0$, then the value of “ a ” is:

(a) between 0 and 1 (b) equal to 1 (c) equal to 2 (d) between 2 and 4 (e) equal to 6

Give reasons for your answer.

10. If $f(x) = \begin{cases} x + 3, & 0 \leq x < 2 \\ 7 - x, & 2 \leq x \leq 4 \end{cases}$ work out the definite integral $\int_0^4 f(x)dx$.

Explain your answer by showing all calculations clearly.

Figure 3. Some of the questions from the post-test.

Results

To test the hypothesis that the effect of the dynamic mathematics software module had produced an improvement in students’ understanding of definite integrals, as measured by the tests, the pre- and post-test results of the 42 participants were compared using a paired t -test, as seen in Table 2. This shows that there was a significant difference between the pre-test mean scores for the two groups, but no significant difference between the post-test mean scores.

Table 2: Analysis of the Experimental and Control Group Mean Test Scores

	Experimental group ($n = 23$)	Control group ($n = 19$)	t	P
Pre-test mean scores (%)	31.9	54.4	3.72	$p < 0.001$
Post-test mean scores (%)	32.4	36.6	0.71	n.s.
Mean difference	0.48	-17.8	-2.91	$p < 0.01$

Considering Figure 4 helps us to see what had happened in the two tests and to make a visual comparison between the scores of the two groups. We can see that the median, quartiles and maximum score all dropped for the the control group, and while the median also dropped for the experimental group it dropped less and the spread is greater, with the upper quartile and maximum mark increasing. The difference in pre-test mean score, in favour of the control group, can be attributed to the differing academic demographics of the two groups, such as their better understanding of the mathematical content taught in the previous year (see above). Furthermore, while most students in the control group were average-achieving students who were progressing well in the Year 13 calculus course, some students in the experimental group were struggling to cope with the content.

The comparisons above suggested that an analysis based on the difference in score between the two tests would be better, and this is also shown in Table 2. Here we see that the post-test scores for the control group dropped significantly, by about 18% on average, whereas the mean scores for the experimental group increased a little, by 0.5%. Hence, the mean improvement of the experimental group scores was significantly greater than that of the control group, confirming that the module most

likely had a positive effect on their learning of Riemann integration. Confirmation of this improvement was sought through formation of matched pairs of students from each group, matching them on the basis of their pre-test scores. It was possible to form eleven such pairs where the pre-test scores were the same, or had a difference of 1 mark, giving an overall mean difference of just 1.2, in favour of the control group (see Table 3). The relatively small number of matched pairs was due to the fact that the pre-test range score of the control group was significantly larger than that of the experimental group ($\text{Range}_{\text{control}} = 86.7\%$ and $\text{Range}_{\text{exp}} = 46.7\%$). Table 3 shows that the results of the experimental students were significantly better than those of the control group, providing further evidence that the use of the dynamic mathematics software GeoGebra may have supported the fostering of versatile mathematical thinking in the students, improving their conceptual understanding of the concept of integral of a function.

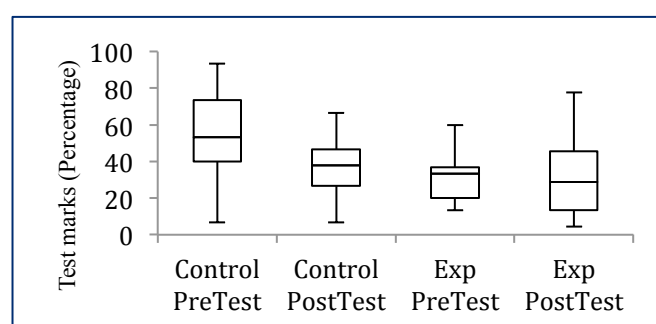


Figure 4: Box plots of pre-test and post-test comparison for each group.

Table 3: Comparison of the Mean Scores of the Eleven Experimental and Control Group Matched Pairs of Students

	Experimental group ($n = 11$)	Control group ($n = 11$)	t	p
Pre-test mean scores (%)	37.6	38.8	-0.18	0.43
Post-test mean scores (%)	43.4	29.9	1.81	$p < 0.05$

Identifying where the gains occurred

An analysis of groups of individual questions was carried out to try and identify where any improvements in understanding had occurred. Questions 2 and 3 (see Figure 3) were designed to investigate students' conceptual understanding of the relationship between the area under the graph of a function and translations of the graph. There was weak evidence of a significant difference between the two groups, in favour of the control group, for a translation parallel to the x -axis (Q2(i) – 11 experimental correct versus 14 controls; $\chi^2 = 2.89, p < 0.1$), but was no evidence that either group performed better on the translation parallel to the y -axis (Q2(ii) – 16 experimental correct versus 10 controls; $\chi^2 = 1.27$, n.s.). The main difference between the groups lay in the method used, with 13 of the experimental group students employing a graphical approach, while none of the control group did. Figure 5 shows a typical graphical response.

In contrast 8 of the controls used an algebraic approach compared with only 2 in the experimental group. This was often of the form $\int_0^4 (f(x) + 3)dx = \int_0^4 f(x)dx + \int_0^4 3 dx = 20 + [3x]_0^4 = 20 + (12 - 0) = 32$. Both approaches are valid and useful,

and one would hope that a student with versatile thinking would be able to solve the problem with either approach. Question 3 was similar to Q2(ii) except that the limits were given algebraically rather than numerically. There was no difference in performance on this question (Q3 – 9 experimental correct versus 5 controls; $\chi^2 = 0.77, n.s.$), with both groups finding this more difficult. One of the students expressed her problem with the question.

Student A: Because I wasn't quite sure if you are writing in terms of a and b , does that just mean you are writing with this [a and b]? Is it just the same as Question 2(ii)? But if it has been translated up by 5, the values on here [a and b] won't change, right? It is difficult for me... Yeah, if it had the values, like 1 and 3, I think I can do it.

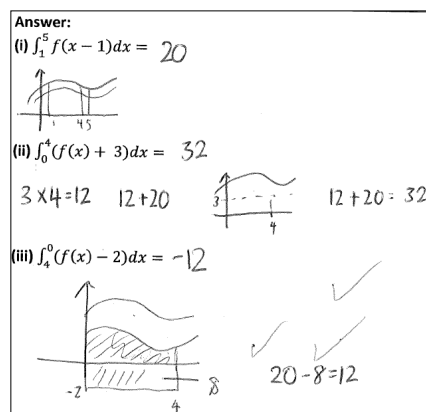


Figure 5: Example of a student's answer (from experimental group) using a graphical approach.

Questions 4 – 6 of the post-test considered whether students had grasped the concept of Riemann sums and whether they were able to use a numerical method to estimate the area under a curve. Table 4 gives the proportions correct for the questions.

Table 4: Proportions of Correct Answers for Questions 4–6 of the Post-Test

Post-test Questions	Proportions of correct answers			
	Experimental group	Control group	z	p
	($n = 23$)	($n = 19$)		
Q4(i): Ability to carry out numerical calculation for Riemann sums	0.09	0	0.64	0.26
Q5: Conceptual understanding of Riemann sums	0.35	0	2.48	<0.01
Q6(i): Understand numerical methods graphically	0.30	0.32	-0.20	0.58
Q6(ii): Ability to use numerical methods to approximate the area under a curve	0.13	0.58	2.75	<0.01

The students in the experimental group performed significantly better than the control group students on Question 5, which considered the process of the area approaching the Riemann limit, and was emphasised in the teaching. One reason that the question was harder was that the graph was not a standard increasing function with positive y -values. One of the students from the experimental group was able to give an

explanation of an increasing number of rectangles in this case.

Student A: I think the answer is more than 7.62. Because if the graph of the function is like this [points to the rectangles under the graph], then more rectangles means the rectangle gets smaller [the width of the rectangle]. So you would get closer to the line [curve of the function], and hence it is more accurate...this one [the left-hand rectangle under an increasing function] is under the graph, so closer to the line and more accurate mean it covers more of the area.

On the other hand, the control group students answered question 6(i), involving trapezium and Simpson rules to find approximations, significantly better. This was not surprising since their teacher spent a considerable amount of time on this topic.

Five questions in the post-test addressed students' understanding of the notion of the "signed area" and the "area-so-far functions". In addition, Question 11 was constructed to see whether students understood the Fundamental Theorem of Calculus. Table 5 shows that students from both groups did not fully grasp the concept of the "signed area", with relatively low numbers of students in each group able to give correct responses to questions 7, 8 and 12. There was weak evidence that the control group students did better in Qs 7 and 8. Very few students (only three in total) were able to give a correct response to question 11. These poor results from both groups indicate that the students still did not fully grasp the Fundamental Theorem of Calculus, and were unable to see the connection between the area-so-far function and the antiderivative.

Table 5: Proportions of Correct Answers for Questions 7–12 of the Post-test

Post-test Questions (abbreviated)	Proportions of correct answers		<i>z</i>	<i>p</i>
	Experimental group	Control group		
	(<i>n</i> = 23)	(<i>n</i> = 19)		
Q7: If $\int_a^5 6x \, dx = 63$, what is <i>a</i> ?	0.13	0.37	1.44	<0.1
Q8: See Figure 3	0.26	0.53	1.44	<0.1
Q9: Sketching the area-so-far function	0.17	0.11	0.19	0.43
Q10: See Figure 3	0.09	0.26	1.11	0.13
Q11: Given that $A(x) = \int_0^x f(t)dt = 0.5x^4 + x^3 - x^2 + 5x$, find <i>f</i> (<i>x</i>).	0.09	0.05	-0.17	0.57
Q12: If $f(x) = x^2$ and $h(x) = x$, then $\int_{-a}^a f(x)dx = ?$ and $\int_{-a}^a h(x)dx = ?$	0.30	0.47	0.81	0.21

Overall the experimental group students commented that the work had helped their understanding.

Student A: Before I think integration just a formula that is opposite to differentiation. But now I know more why you do integration, like it is for finding out the area. Basically I see more connection between differentiation and integration now.

Student M: I think my knowledge of the relationship between integration and area has changed. It [GeoGebra] probably enhances my knowledge of area and improves it. I mean before I see integration just as a technique or a formula opposite to

differentiation. But now after you [the teacher] have explained to us with the computer program, I understand more why we use integration for area.

Student E: At Year 12, I just think integration as a simple algebra algorithm. Now after you taught us the topic with the computer program, I think I have learned more about integration such as how it relates to the area. I understand it better graphically now.

One student (A) from the experimental group obtained 13.3% on the pre-test but scored 48.9% on the post-test, improving by 35%, and placing her among the top four students for improvement in the experimental group. She had gained significant improvements in understanding in some conceptual areas, especially in: the relationship between area and the definite integral of a function crossing the x -axis; and effects of transformations parallel to the x - and y -axes. In her interview she was also able to provide correct responses for the three questions about the relationship between area and the definite integral. While she had a weak algebra background, making it difficult for her to develop versatile thinking fully, the technology may have enhanced her thinking about graphical aspects of the content.

When they were asked in the interviews for their views on the value of GeoGebra in learning they all mentioned that the visual aspects of the software helped them understand the integration topic better. In particular, it assisted with the learning of the concept of area approximation and how it relates to the definite integral graphically. Some of these comments were:

Student A: I like that you got a visual and you can see what actually you are trying to learn. It just generally helps me to see the concept of integration and also being able to apply it.

Student I: I think the visual display and the dynamic image of the program help my understanding of seeing where the formula is derived from. By looking at the different shapes, rectangles and trapeziums, I understand better how we found the area using different shapes and how it is related to the definite integral.

Student M: I really like it [GeoGebra] when we were learning about the area approximation using rectangle and also trapezium. I can see it on the graphics view instantly that the approximation is better when you [the teacher] started changing the number of rectangles or trapeziums. I guess it is easier for me to understand because it is not just graphical, it is also instant.

Conclusion and Recommendations

Overall the results of this research showed that the mean improvement in the experimental group test scores was significantly greater than that of the control group, with half of the experimental students gaining more marks in the post-test compared with less than a quarter of the control group. Thus it appears that the module based on the use of GeoGebra was a useful addition to the module on integration. When we look at the results in detail it seems that students tend to do better on the topics that were emphasised by their teacher. This is not surprising and it is what every teacher would hope, that the students learn what is taught. However, some previous research [18, 19, 20] has shown that students who were taught with traditional methods have a tendency to see integral calculus as a series of algebraic procedures or algorithms. This seems to have been the case for the control group in this research study. What the experimental group seem to have gained from the computer work was a richer concept image through the addition of embodied, visual thinking, and an ability to

link this to numerical approximations. This is valuable for versatile thinking, but should not be added at the expense of symbolic algebraic work, since both aspects are important. Given that dynamic computer visualisation may support the understanding of connections between graphical and symbolic representations, we believe that allowing the students time to engage in active learning, experimenting themselves with the technology through an experiments (student-centred) approach [21], would add benefit to the module employed here. However, using technology to promote epistemic mediation of mathematical concepts will require teachers and lecturers to develop what Thomas [22,23] calls *pedagogical technology knowledge* (PTK). With this there is a shift in focus, from seeing the technology as simply something added to the teaching of mathematics, to putting the mathematics technology at the centre of learning activities. Evidence is this PTK is enabled by a strong mathematical content knowledge, a positive attitude toward technology, personal confidence in teaching with it, and good instrumentation and instrumentalisation, but its acquisition requires explicit attention through continued professional development.

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Incorporating Student Response Systems in Mathematics Classes

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In the past decade, Student Response Systems (SRS) have been used more widely in higher education as advancement in technology makes them more affordable, easier to use and of compact design. This technology shows potential in enhancing the student experience, especially in traditionally hard subjects like mathematics. Major reasons for introducing the technology into classrooms include positive student perception, anonymity, active teaching and learning and providing a natural break from straight content delivery. There remain issues related to the integration of the technology within mathematics courses - in particular related to the time required during lectures to successfully embed SRS without impinging too greatly on the delivery of the syllabus. In this study the appropriate number of questions that should be posed during a lecture is investigated, as is the time interval that should be permitted per question and when each question is posed. It has been suggested that SRS are not utilized effectively unless questions used provoke deep learning but this can be problematic and impact on endemic math anxiety regardless of anonymity. It is proposed that questions involving higher order thinking may be better explored within a tutorial environment using a Team Base Learning approach.

Keywords: SRS, technology, clickers,

Introduction

Student response systems (also called audience or personal response systems) involve use by most or all students responding directly in class to short questions posed to determine conceptual understanding. These allow the lecturer or tutor to gauge the effectiveness of learning transfer in a class. Commonly students use small input keypad devices ('clickers'; 'zappers') to signal support for an option among those presented to them with a question stem. Multi-choice and true false questions are usually posed – sometimes mixed together – and displayed by prepared slide. The responses are automatically collected and the results analysed in spreadsheet and displayed for further discussion by the class.

The idea has been around for many years (Horowitz, 2006) but only recently have systems been sufficiently affordable and reliable to be widely used. Changes to technology also made systems compact, portable, wireless and easy to use. Even the wireless systems have undergone changes from infrared use to radio frequencies to increase efficiency and reduce collection errors. The explosion in their use in general over the past decade is charted in survey papers such as those by Kay and LeSage (2009) and Caldwell (2007). Retkute (2009) examines their use in mathematical

¹ Part of this work was done while this author was on leave at the School of Arts and Sciences (Vic), Australian Catholic University

sciences in particular. Titman and Lancaster (2011) report that one expert claimed some use of SRS existed in almost every university in the USA by 2006. The principal use has been in the science disciplines but there has been use in almost every type of course. The use varies from large lectures to small tutorials. It is used where class styles reflect classical instruction with interposed questions used to check student progress or attempts to run classes more as dialogues on material that students are expected to read in advance.

In this work we will briefly review the nature of and the reasons for using student response systems and examine some of the issues affecting its use, both technical and pedagogical. We will then analyse the outcome of using the SRS over a period of 3 semesters in classrooms where mathematics was taught in three different subjects to pre-service teachers in first and third year of their undergraduate course, and discuss the results in terms of prevailing theory and practice. The same questions used in two subjects provide for an analysis of response times and surveys on students in all three semesters provide information on attitudes and use.

Motivation for Using a Student Response System

Some educators claim the most common and traditional didactic delivery method in mathematics education is often seen as too disengaging, ineffective and therefore inefficient for large student numbers (Larsen, 2006). Whether we agree with this completely, it is clear that students are frequently absent physically from classes for many reasons including false perception of ability, excess outside social life and work and the tendency for mass education to funnel people into courses they may not be suited to or really seek. Students can readily feel isolated in large classes. This would be serious enough for subjects which are central to student interests but mathematics classes at university are often taken as service subjects, a major component of a course but not central to the student interest. A student's need for mathematics may then not be naturally coupled to high ability in the subject which can lead to student negative perception that mathematics is too hard and boring – reinforced by mathematics anxiety and phobia.

A number of benefits are claimed by advocates of these student response systems. We consider these in turn. The systems are automated and provide immediate compiling of responses. Response frequency data can be displayed to the class for discussion so that the various choices can be explored further including the opportunity to correct widely held misconceptions if this is needed. One common criticism observed in classes over time is the issue of poor feedback for students and this can be overcome by this process as each student gets a chance to vote and receive some feedback on their own opinion. Moreover this vote can be anonymous – a major issue in favour of the systems. Asking students to raise hands or comment on an issue directly does not allow effective and efficient data collection and display and it invites poor response rates or responses affected by student perception of their peers' opinions (conformity). This issue is well documented by Stowell et al. (2010) where a comparison of traditional approaches with clicker systems was outlined. The student response systems improved validity of responses by increasing variety and reducing conformity and boosted participation as anonymity ensured reluctance to be noticed at all or seen to be wrong was overcome. This feature is highlighted in most papers discussing the use of clickers and remarked on in survey papers (Caldwell, 2007; Kay and LeSage, 2009).

Favourable student reaction (which we see is admittedly not universal – see figure 2) to use of the response systems has been identified as a cause of improving student

engagement with the class (on an individual basis) and for improving classroom attendance overall with the prospect that time on task will be improved – a key feature in the learning process. The prospect of improving engagement in class is remarked by Titman and Lancaster (2011) where this is believed to link directly to good pedagogy. This automatic link is questioned by Dangel and Wang (2008) who caution about the quality of engagement being a determinant on good learning. This issue of classroom attendance is also reported by Titman and Lancaster (2011) where they note poor attendance in class is discussed in several papers as a motive for introducing the clicker systems. It is addressed in both major surveys (Caldwell, 2007; Kay and LeSage, 2009). In some cases, this increased participation may be dependent on related variables. In a large engineering mathematics class, use of the clicker technology by d’Inverno et al. (2003) was accompanied by a use of ‘skeletal notes’ where students received partially completed notes which they needed to finish in class. The ostensible reason for this was to speed the flow of class presentation and allow time for clicker use without the commonly reported reduction in coverage of material. However students are more likely to attend regardless, if the subject notes sold contain gaps to complete in class, in contrast with courses where notes supplied are complete. The effects confound one another at this level.

An additional advantage is reported in promotion of team-based learning (Haeusler and Lozanovski, 2010). In this application the formation of small groups after an initial question session allows students to reflect on the results and review their opinions. This may address concerns by Dangel and Wang (2008) about the style of teaching which is reinforced by common clicker usage but it may seem to obviate advantages emphasised by Stowell et al. (2010) on the benefits of anonymity in suppressing conformity. In this case however a preliminary use ensures the legitimate response and the small group discussion forms part of the proper learning process. In any subject like mathematics where there is a true answer, the aim is for all students to conform to that opinion finally!

Methodology, Implementation and Practicalities

Three mathematics education courses were selected for this study which covered a period when mathematics education subjects at USQ [Faculty of Education] were re-vamped. The first course, in semester 1 of 2010, was the fourth year² mathematics course named *Becoming Numerate* that covered all the five strands of mathematics education. This was the last semester the course was delivered in favour of two re-vamped courses, namely the first year course *Introduction to Numeracy*, in semester 2 of 2010, and *Mathematics Pedagogy and Curriculum* in semester 1 of 2011. Broadly speaking, the sum of the content of these two re-vamped courses was equivalent to the former course, *Becoming Numerate*. Attendance is not assessed in itself. Approximately half the students attended in the clicker classes. In two subjects the material was also supplied online which can affect attendance. This attendance rate is in line with proportions reported in the literature and the increasing of this attendance and engagement of students is one motive for incorporating SRS in a course.

In all courses in the study, students had a traditional lecture and a two hour tutorial. Upon entering the lecture or tutorial room’s students were instructed to pick up their “clicker” for use in class – each student was assigned a clicker number for the entire semesters work. A clicker is a radio transmitter with an electronic key-pad

² Please Note that “fourth year” here does not imply fourth year level content but merely that the course was available only to fourth years students enrolled in education courses.

(resembling the key-pad on a mobile phone) where the signal is received by a (compact) USB Dongle. No other hardware component is needed and the system can be used on any PC or Laptop where the proprietary software is installed. The software is an add-on within Microsoft PowerPoint. In lectures and tutorials students would be asked a series of multiple-choice and/or true-false questions. The lecturer would first read the question to the class and the possible answers (in the case of multiple-choice questions). Students would then respond to the question by depressing a number on their key-pad corresponding to the choice they assumed was correct. Then the overall response to a question was displayed on a bar-chart for the entire class to review. Finally, a classroom discussion would then address any misconceptions. The rate of giving questions was three per class-hour. A sample set used for analysis here are given in the Appendix.

In the event that close proximity of classes using the clickers should lead to interference it is easy to set the frequency in each room to a different level. Otherwise, the novice user may be surprised to see data arrive on the unit before they gather it which is merely a technical error!

In the first course that the clickers were introduced all questions were posed after the delivery of the course content. In this case, the clickers effectively facilitated an end-of-lecture/tutorial formative quiz. However, this approach clearly did not take full advantage of the technology being used. The clickers offer a chance to break lecture flow at natural concentration walls and may be better used this way. The time used in running the SRS comes from teaching time. This raises a number of fundamental questions – what is the optimal number of questions that should be posed during a lecture, what time interval should be permitted per question and when should each question be posed? Moreover, the types of questions posed are of critical importance as higher order questions (questions that required high order thinking), although needed, proved a stumbling block as far as the practical application of the program.

Table 1. Means and Medians for Survival (Response) Time

Question No	Mean ^a				Median			
	Estimate	Std. Error	95% Confidence Interval		Estimate	Std. Error	95% Confidence Interval	
			Interval				Interval	
			Lower Bound	Upper Bound			Lower Bound	Upper Bound
1	4.729	0.373	3.998	5.460	4.454	0.106	4.246	4.661
2	11.266	0.612	10.066	12.465	11.275	0.231	10.823	11.727
3	64.923	3.816	57.443	72.403	49.327	5.542	38.464	60.190
Overall	26.973	2.363	22.341	31.604	11.275	1.450	8.433	14.117

a. Estimation is limited to the largest survival time if it is censored.

The student response times were studied in order to address the issue of the optimal number of questions and the time dedicated to each question. This was only applied to two of the courses as a sub-set of students from the latter went onto the third course, and this removed repeated exposure. The probability that a student will take longer than a specified time to respond produces an empirical distribution. Maximum likelihood estimators for this yield the Kaplan-Meier estimates from classical survival data analysis. These were produced for the response times with the

analysis conducted using SPSS (see eg. Titman and Lancaster, 2011). This generates confidence intervals on ‘response’ times’ – the time to final response or allows censored data to be used where the response was timed out. The estimates generate a step graph as shown in figure 1 and also give measures of location for response times as shown in table 1.

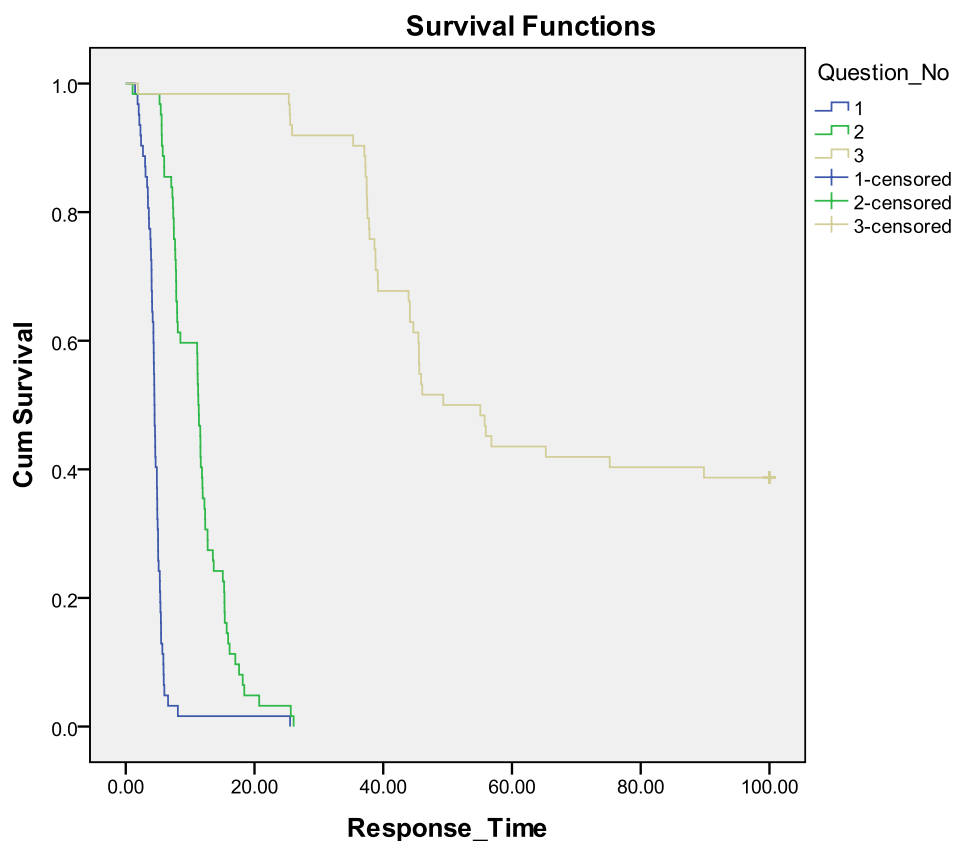


Figure 1. Graph of cumulative response times.

A comparison of the response time data between questions is interesting and instructive, as the graph in figure 1 shows. The first question was of no issue at all with over 95% of students responding within 10 seconds (after the question and possible responses was read out loud). This was reinforced with a 90% correct response to Question 1 (see appendix). It is interesting to note that a few students either responded immediately or took considerably longer to respond relative to their peers. Several reasons may explain this common phenomenon – a technical or human error, or perhaps a manifestation due to student’s objection to the use of the technology (D’Inverno et al, 2003). The second question proved more of a challenge as 95% of students responded within 20 seconds, but only 50% with a correct response. This question required more classroom discussion than the first. Again, some students either responded immediately or took considerably longer. The last question was the most problematic as 39% of student could not respond within a reasonable time (100 seconds) and of the 61% that did reply within a reasonable time only 37% responded correctly. It is evident that Question 3 requires substantially more computation than Question 2 which the students could not deal with in the lecture environment. Although a classroom discussion was then initiated, this problem was a good candidate for further and more detailed consideration in a tutorial setting. This information is still useful to the lecturer as it shows that more time on the idea is

needed for learning to occur.

The output in table 2 shows that the response times could not reasonably come from a single distribution pattern which is also strongly suggested by figure 1. Although the students only receive 5 options and can press a clicker in the same time regardless of question level, they clearly choose to reflect on a response and not merely give a random or first guess. This is despite the total anonymity involved. Response time patterns will differ with level of difficulty even though the correct response is always a selection of a listed option.

Table 2. Statistics on Response Time Pattern

Overall Comparisons			
	Chi-Square	df	Sig.
Log Rank (Mantel-Cox)	284.668	2	0.000
Breslow (Generalized Wilcoxon)	227.920	2	0.000

Test of equality of survival distributions for the different levels of Question_No.

Student Perception and Benefits

Student perception of SRS use has been commonly found to be very positive as noted by Kay and LeSage (2009). This is confirmed by a survey carried out on subjects over 3 semesters at USQ. Students were surveyed on seven items and the 5 point Likert scale of agreement was analysed to show the accompanying bar-charts in figure 2.

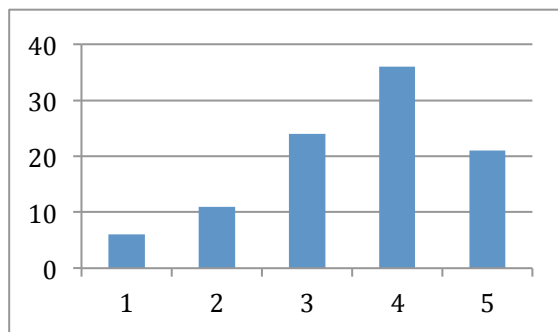
The mathematics education students reveal the method is generally popular but as noted in other institutions there are detractors and here they form up to 20% of the cohort. The negative responses were most marked on the issues of whether the clickers assisted learning, made the class more engaging or clarified the student how they stood with regard to their peers. It is interesting to note that some students who claimed it did not assist learning clearly still believed it helped them identify areas of strengths and weakness. This may imply they still were unable to decide how to get help to rectify the weaknesses or capitalise on their strengths and this in turn means it may be the post- SRS discussions that need examination.

Discussion

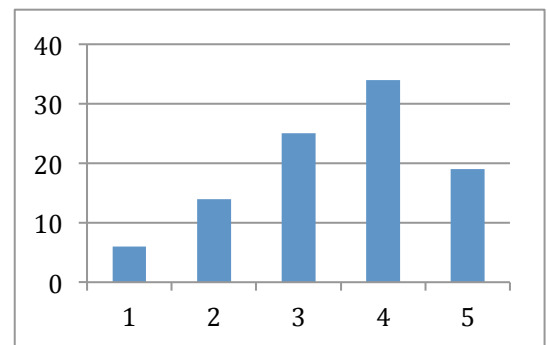
Rapid change to secondary level delivery of mathematics, incorporating use of technology, can promote high expectations in students entering tertiary education (d'Inverno et al, 2003). It would be folly to use technology for its own sake or misuse it but students are now clients and expect universities to invest in constant improvement in course development and delivery and the university system is a free market. If more students like using clicker technology than not, and feel more engaged regardless of final performance it creates pressure to provide this. Many authors argue this fact and our results confirm it as seen in figure 2.

If we are to use an SRS it is important it be used wisely as noted by Beatty (2004) and emphasised by Dangel and Wang (2008). Beatty (2004) talks of transforming student learning with clicker technology but remarks that the form of questions used need to have a clear and defensible pedagogic goal, and he lists seven examples of these which draw out understanding, highlight and remove common misconceptions and do more than probe memory or call on facts or computation. Dangel and Wang (2008) suggest that their survey reveals little of this in the visible literature painting a depressing view that the clicker technology is not definitely associated with any

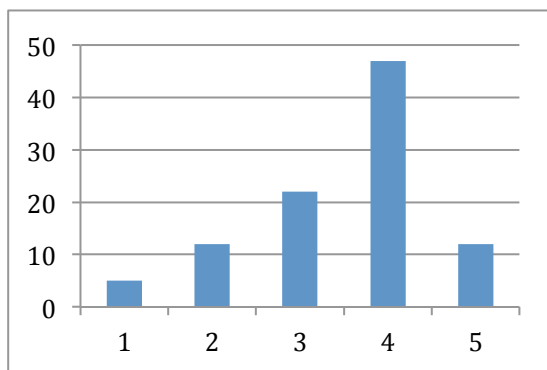
measurable improvement in overall performance which they attribute to its use in surface learning.



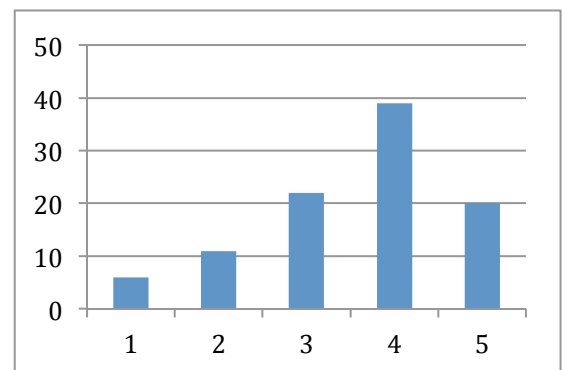
Was not a waste of time



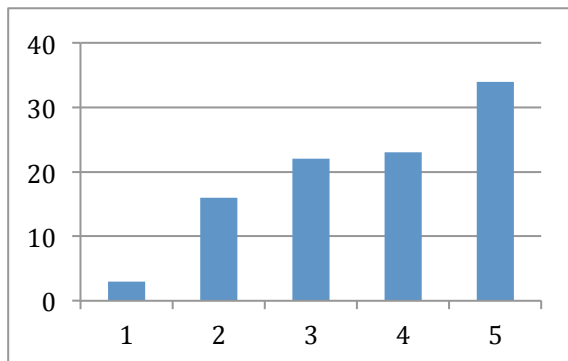
Did assist my learning



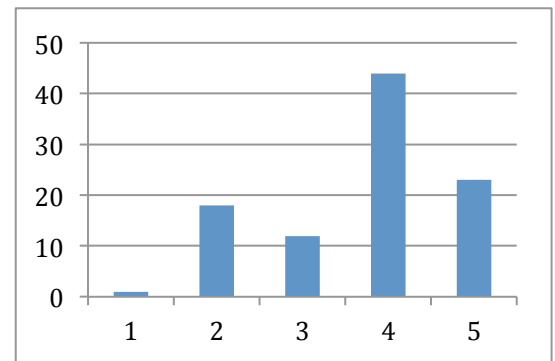
Assisted understanding of concepts discussed



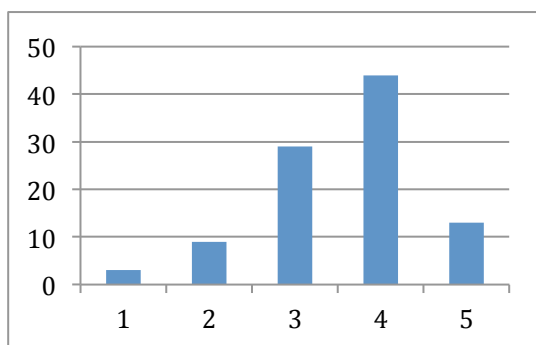
Prompted discussions in my group



Made the classes interesting and engaging



Showed me where I stood in relation to the rest of class



Helped me identify areas of strength and weakness in content knowledge

Figure 2. Bar charts on student attitudes (1 = strongly disagree 5 = strongly agree).

We have seen that students generally do not provide answers indifferently, and this has implications for the recommendation of Dangel and Wang (2008) that only deep learning questions, which take longer to reply, should be used. They rightly urge that the SRS be married to good pedagogy to enhance deep learning rather than decry the technology however. This data on student response times does argue against questions that need any serious calculation, which may be merely time consuming but not insightful in pedagogical terms.

In the first question which was used in the sample discussed in this study the aim was to use the SRS to assist in engaging student attention and reducing math anxiety which was endemic to the cohort. Where this is the aim, the question needs to be posed at an easy level. The issue of pedagogy is strained further when we see that the third question used had many non-respondents as students could not answer it, and declined to 'guess'. This could signal more time needed to be allocated but it also brings up one significant issue with the clicker systems.

One serious problem identified in the literature is that use of an SRS removes teaching time. The time taken includes reading out questions, waiting for longest response time (or choosing a cut-off) and clarifying any misconceptions found. The time sacrificed in this sample was about 3 minutes of waiting time, about 1 minute of reading time and perhaps 4 minutes of discussion in a 50 minute class totalling about 16% and this reduction is reported to be of the order of 15% to 30% in different studies. As d'Inverno et al. (2003) note, this is an unacceptable loss of coverage for providers of a service mathematics course – even if we could demonstrate that students were learning better and retaining more. Courses are already under twin pressures of coping with entry students of reducing background and course management pressure on time allocated to service subjects. This is certainly the case in engineering mathematics for example. Other disadvantages can be identified to use of an SRS. The technology itself carries a cost and whether this is borne entirely by the institution as at USQ or partly by students as reported in some US universities, it has implications for use. Some students are not engaged by the SRS and have negative reaction to it, as seen in our survey in figure2. This may be exacerbated if universities elect to make students pay for their own clicker as occurs commonly overseas.

In summary, the use of an SRS comes down to balance in a gain/loss case. The gains include frequent nonthreatening feedback for students on learning and for lecturers on student understanding. This in turn offers a chance for better learning transfer. The overall student favourable perception of SRS shown here and in the literature offers a prospect of better engagement by those attending and higher attendance in classes. The losses include a need to reduce class coverage of material through time used in the SRS and the initial cost of implementing the process.

This issue of time used on the SRS needs to be addressed. Either more time in the syllabus needs to be made by using other devices like skeletal notes and/or technology to assist coverage like use of CAS or we need to find an acceptable loss level at which to operate. The amount of time we could sacrifice will dictate how many questions can be used and how long they may take. In practice the number of questions set here also assists in breaking up an hour into more manageable periods of concentration for students. The level of questions should vary and the final question here showed that anxiety may remain even with anonymity which may argue against using each question to probe deep learning; some may be chosen to just inspire participation. There is no clear answer on how many questions can be used in a class although more than 3 or 4 seems unnecessary, and the time loss would be too high. However the

issue on how to gain more effective learning transfer as identified by Dangel and Wang (2008) remains unresolved and so we need to examine cases where the technology was used and it lead to improved results. The items students traditionally have trouble with are always a good start for building suitable questions.

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Appendix

The multiple choice questions chosen for the analysis:

Question 1: How many Tens are there in 13201...?

1. Zero
2. 20
3. Three hundred and twenty
4. 1320
5. 132

Question 2: $1/(5 \times 5)$ has the equivalent decimal form of...?

1. 0.5
2. 0.05
3. 0.2
4. 0.4
5. 0.04

Question 3: The Base 10 value of 444_5 is...?

1. 124
2. 444
3. 421
4. 123
5. 4440

Mathematics and its Connections With Technical Subjects, the Real World and Professional Life

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This paper will offer an analysis from a theoretical point of view of mathematical modelling and its applications. We will also analyse the inverse problems, of both causation and specification types. After that, experiences from courses taught over the past 15 years will be described and analysed. A separate section will briefly describe a particular experience carried out with former students, dealing with problems arising from the industrial sector. On the basis of this experience, the characteristics of real-world problems will be defined. Finally, several results will be presented and some conclusions proposed.

Keywords: modelling and applications, inverse problems, teaching experiences.

Introduction

Basic sciences, especially mathematics, provide the basis, the language, and so on, for technical subjects. However, mathematics teachers often have the feeling that their subject is underutilized in higher education courses. For example, the "inexact differentials" that are so common in Physical Chemistry and/or Thermodynamics textbooks, and that could be avoided by using the basic ideas of Vector Calculus [1]. Another instance is the Gibbs' Phase Rule, which in many textbooks is arrived at by pseudo-deduction, counting equations and unknowns as if they were a linear system of independent linear equations, when it could be correctly arrived at using Lagrange Multipliers [2].

At other times, mathematical ideas and concepts seem to clash with certain real life situations. For example, if a fluid is flowing along a straight tube in a purely laminar flow, with a total absence of turbulence, there would appear to be no rotation and no vorticity; however, curl is non-zero at every point, except at the centre of the tube [3]. At first sight, at least, this appears to contradict what is taught in mathematics courses, where the rotor is an indicator of the gyrational tendency of the vector field.

This apparent contradiction between basic subjects and technical subjects is not helped very much by the customary approach of traditional mathematics courses. In fact, these courses and the ways in which they are assessed emphasise routine exercises that encourage acquisition of memory skills and rigid procedures; at best, if problems are set, they tend to be self-contained direct problems [4]. Very rarely are inverse problems posed, and those few that do make an appearance, do not involve mathematical modelling nor applications to real life situations. In actual fact, the problems that arise in real life are quite the opposite to the exercises worked on in traditional courses: they are frequently inverse problems, and they also tend to require mathematical modelling skills. We have chosen to call these inverse problems, for which mathematical modelling is essential, "*Inverse Modelling Problems*" [5], and they are the type that most commonly occur in professional practice, as we shall see

later on in this article.

This paper will offer an analysis from a theoretical point of view of mathematical modelling and its applications. We will also analyse the inverse problems, of both causation and specification types. After that, experiences from courses taught over the past 15 years will be described and analysed. A separate section will briefly describe a particular experience carried out with former students, dealing with problems arising from the industrial sector. On the basis of this experience, the characteristics of real-world problems will be defined. Finally, results will be presented and some conclusions proposed.

Theoretical Framework

Inverse problems

Direct problems, according to Groetsch [4], can be seen as those that provide the necessary information to follow a well-defined, stable procedure leading to a single correct solution.

Inverse problems, in contrast, are both more difficult and more interesting, largely due to their either having multiple solutions, or not being capable of being solved at all [6]. They frequently crop up in the practice of several professions and careers. For instance, when treating a patient for a particular illness, listing the symptoms is a simple, direct problem that has already been solved and can be looked up in any medical textbook. On the other hand, diagnosing a patient's illness from his or her symptoms is not always a straightforward task, and requires an experienced doctor.

Inverse problems have not always been properly studied, and a quote from Bunge [6] is appropriate: "The fact that almost all philosophers have ignored the peculiarities of inverse problems poses another inverse problem: to guess the reasons for this huge oversight on the part of philosophers."

In principle there are two different types of inverse problems, but in order to characterise them correctly, let us begin with a schematization of direct problems, adapting a study by Groetsch [4]. His scheme for a direct problem is like this:

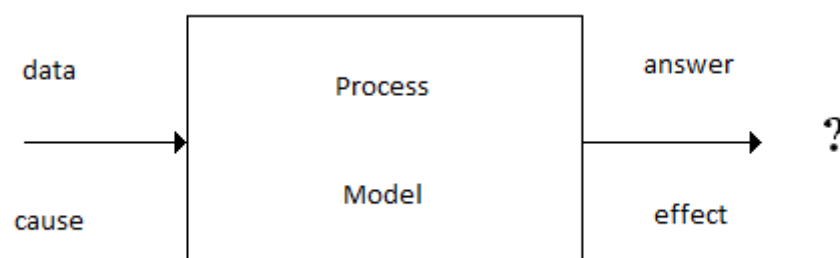


Figure 1. Schema of direct problems.

In the scheme in Figure 1, data and a given procedure are available, and the answer is sought; for instance two polynomials are given, the division algorithm is known, and the answers required are the quotient and remainder polynomials.

Now we can change the schema to obtain two inverse problems. The first is the causation problem, schematized in Figure 2:

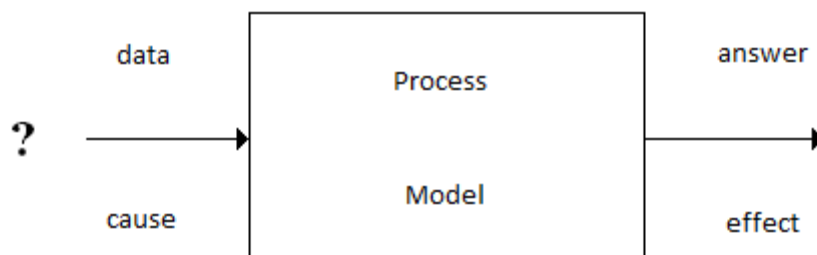


Figure 2. Schema for causation problems.

The other inverse problem generally encountered is the specification problem, schematized in Figure 3:

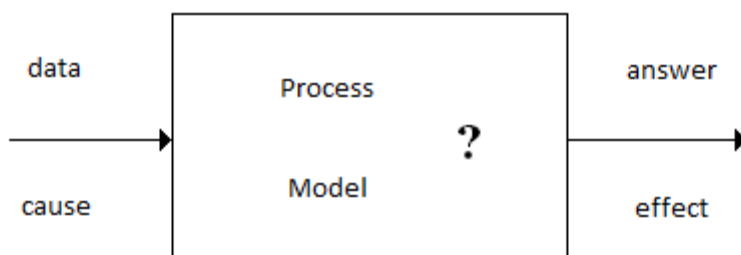


Figure 3. Schema of specification problems.

Both types of problems are common in the experimental sciences. For example, in university chemistry courses [7], in Qualitative Chemical Analysis classes students are often given a test tube with a solution containing three or more unknown cations, which students must identify from the results of a particular sequence of reactions to test the problem sample. This is, therefore, a typical causation problem.

In Inorganic Chemistry, on the other hand, the problem is more frequently to synthesize a given salt from simpler substances, and therefore the problem does not lie in identifying the reactants and/or products, but in knowing and correctly carrying out the process that will lead to the formation of the desired product, starting from cheap and easily available reagents. This is clearly a specification problem [7].

Finally, in Organic Chemistry both inverse problems (causation and specification) tend to be posed simultaneously, for instance during "organic synthesis", when reagents and processes must be determined simultaneously [7].

Modelling and applications

Problem solving, modelling and applications are not synonymous, although they are obviously related. For instance, the Discussion Document preparatory to the ICMI Study 14 [8], mentions that the term "modelling" focuses in the direction that goes from the real world towards mathematics, while the term "applications" goes in the opposite direction, that is, from mathematics towards the real world. In addition, the term "modelling" emphasises the process that is taking place, while the word "applications" stresses the object involved (particularly in areas of the real world that are susceptible to a given mathematical treatment). The same document uses the term "problem" in a broad sense, including, therefore, not only practical problems, but also abstract problems, or those that attempt to describe, explain, understand, or even design parts of the real world.

Obviously, solving problems and modelling are not the same thing (one can

model even in the absence of a concrete problem to be solved; one would simply be giving a mathematical description of a given phenomenon), and of course, "pure" mathematics problems can be solved, which do not require any kind of modelling.

In the light of the above, we can arrive at the following schema:

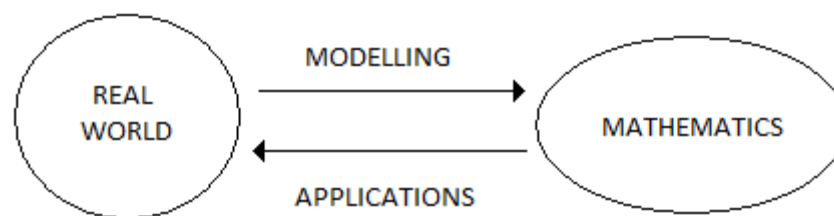


Figure 4. Scheme of modelling and applications.

Finally, it is worth mentioning that a more extensive discussion about modelling, applications and problem solving, and their teaching on university courses and secondary education in Latin America is available in a previous paper [9].

Teaching Experiences in Different Mathematics Courses

Our first experience of teaching about inverse problems and modelling was in 1996, on a course called Mathematics III, basically organised for Food Engineering students, which was followed by Mathematics 005 and Mathematics 105, (within the revised course plan of the year 2000) that were basically designed for students of Food Engineering and Chemical Engineering, respectively.

The students on these courses were all undergraduate students on different Chemistry degrees, basically Chemical Engineering and Food Engineering, although there were a few studying Pharmaceutical Chemistry, and Chemistry. The degree courses are quite different, but the students all had one thing in common: they were all used to tackling inverse problems in other subjects. Specifically, in Analytical Chemistry they came across causation problems; in Inorganic Chemistry they encountered specification problems; and in Organic Chemistry they came across both (see [7] for more details). In contrast, they had virtually no previous experience of modelling, as although some applications had been taught on previous courses (such as Calculus and Algebra, prior to 1996), they had not come across real modelling problems. On the contrary, in the first half of the 1990s, the dynamics and style of classes, assessments, exercises and so on were quite traditional.

As a result, mathematical modelling was taught right from the first year of the course (i.e. 1996), in order to correct the main gap in the students' knowledge at that time. In subsequent years, inverse problems were included: causation problems in 1997, specification problems in 1998, and from 1999 all types of inverse problems (causation, specification and mixed) were studied, using modelling and mathematical applications skills and tasks, simultaneously.

In the second semester of 2002, Uruguay was hit by the worst economic crisis in its history [10]. Some teachers left the country, others took up more lucrative occupations, and the project, at least at the undergraduate level, collapsed because of the lack of a "critical mass" of human resources.

Fortunately, before the crisis of 2002 courses for other teachers and/or researchers were already under way, including a postgraduate course in 1997, several Permanent Education courses between 1997 and 2000, and an initial course abroad, in Buenos

Aires, in 2001. After the crisis events in 2002, further courses were taught abroad (in San Juan, Argentina in 2005; in Tucumán, Argentina in 2009; and in Guatemala City, Guatemala in 2010), which had good results and were well-received (see, for example, [5-7-9]).

Returning to Uruguay, teaching was resumed – although not exactly at undergraduate level – with a job at the Julio Ricaldoni Foundation (JRF), which will be partly described in the following section. Over and above the importance that working for JRF has in terms of industrial consultancy, it is also an unit for the training of human resources, in that advanced undergraduate students participate in the projects.

An Unusual Experience

In the second half of 2009 a working group was formed at the Julio Ricaldoni Foundation (JRF), made up of a former postgraduate student in Chemical Engineering with a Master's degree in Corrosion Engineering, two undergraduate students and the author of the present paper. The group's aim is to try to solve a problem affecting the main electricity generating plant in Montevideo, the Central Batlle, in particular the refrigeration circuit, which is showing signs of advanced corrosion in the heat exchanger pipes.

A first approach of the problem is the following one:

$$\left\{ \begin{array}{l} \frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} + \frac{\partial^2 E}{\partial z^2} = 0 \\ \text{with :} \\ \frac{\partial E}{\partial r} = 0 \quad \text{when} \quad r = 0 \\ \frac{\partial E}{\partial r} = K(E - A) \quad \text{when} \quad r = R \\ E = \Phi_1 \quad \text{when} \quad r = R \quad \text{and} \quad z = 0 \\ E = \Phi_2 \quad \text{when} \quad r = R \quad \text{and} \quad z = L \end{array} \right.$$

In these equations r is the radial coordinate; z is the longitudinal coordinate and E is the electric potential, K and A are constants empirically obtained from a polarisation curve, R is the radius and L is the length of the tube and finally, Φ_1 and Φ_2 are the potentials at the tube edges [11].

A way of simplifying the problem is to assume the electrolyte potential varies only in one dimension, the direction of flow. This concept is owed to Frumkin [12] and can be used, to simplify the problem as follows:

$$\left\{ \begin{array}{l} \frac{\partial^2 E}{\partial z^2} = \frac{2\rho i}{r} \\ \text{with :} \\ \frac{\partial E}{\partial r} = -\exp\left(\frac{E - A'}{B'}\right) \quad \text{when} \quad r = R \\ E = \Phi_1 \quad \text{when} \quad r = R \quad \text{and} \quad z = 0 \\ E = \Phi_2 \quad \text{when} \quad r = R \quad \text{and} \quad z = L \end{array} \right.$$

In this case, the polarisation curve is approximated by a semi-logarithmic curve,

being A' and B' the corresponding constants. Finally, ρ is the resistivity of the electrolyte and i is the circulating current.

This problem can be approached in at least five different ways:

- Solve the original PDE problem numerically, using finite difference methods [13].
- Discretize the radial derivatives in the PDE problem, giving rise to an ODE system that is second-order in terms of z (the longitudinal positional variable), and then change the variable $Y = dE/dz$, resulting in a first-order linear system that can be solved using the exponential of the system matrix. This semi-numerical method is described in detail in [14].
- Divide the tube into small longitudinal sections, and linearize the dependence of current and potential in each of the resulting sections. If the simplified model using Frumkin's condition is used, we obtain a linear ODE that can be solved analytically for each section. This piecewise semi-analytical solution is described in detail in a previous paper [15].
- It is possible to find solutions by wholly analytical means for the simplified problem, once the Frumkin condition is verified. In fact, it can be solved by tackling a simpler, first-order ODE, and then adapting the resulting solution – by means of a variable change – so as to solve the second-order equation. While this approach is ingenious and gives rise to a plethora of solutions [16], there is no guarantee of obtaining all the possible solutions of the problem.
- Analytical solutions to the ODE can also be found by using pre-integrals, or functions that remain constant within the ODE trajectories. This approach can also produce a plethora of solutions, which correspond to some of the cases analysed in the previous option [11] but again, it does not guarantee finding all the possible solutions of the problem.

All these approaches are ways of tackling the direct problem (that is, finding the potential at each point of the tube for given boundary conditions), but really the problem is the inverse: in other words, finding the boundary conditions so that the potential at all points of the tube is lower than the corrosion potential, or at least, so that corrosion of the tube is minimised as much as possible.

This is the normal situation in Engineering: one knows where one wishes to arrive at, but the question is, how? In other words, one knows the desired outcome, but usually one does not have a clear idea of where to start from (a causation problem), nor of what would be the best way to proceed (a specification problem). This leads to one, or both, types of inverse problems.

We should probably ask ourselves whether it is not a waste of time to seek out five or more different techniques for solving the direct problem, when the real problem is an inverse one. The answer to this question is related with the techniques recommended for solving inverse problems [6], one of which suggests studying the family of direct problems, because one or other of these generally holds the key to solving the inverse problem [6-7].

Several Results

Many of the statistical results from surveys about these courses have already been published (see [7] and [9]). However, students' comments in response to some open and semi-open questionnaires have not. Here, the students give their views about the courses, and particularly about inverse problems and mathematical modelling and applications. Some of the most interesting comments were as follows:

Now I find that mathematics can be useful.

A really super course, I got a great deal out of it.

An interesting course, with quite a lot of applications in real life.

Specifically, about tasks related to modelling and applications, they had this to say:

If these topics were omitted, the course would just be another standard maths course, a 'hard' subject filled only with methods, calculations and numbers.

Once the new study plan was instituted, the course known as Mathematics III was replaced with Mathematics 005 and 105. After these changes, students were again consulted by means of semi-open questionnaires. These are some of their views:

Very directly applicable to my undergraduate professional career; it renewed a taste for mathematics and it was well taught, guiding the solutions to the exercises and not working them all out.

I thought the course was very useful and dynamic, and I think it will have very useful applications in coming years.

As for the modelling, applications, and the problems set (direct and inverse), these were some of their comments:

The problems were motivational because you can see the usefulness of mathematics in daily life, and they clearly show the interaction that exists with other subjects.

I have taken courses in which the applications dealt with here were relevant.

Very useful examples for future years of undergraduate study.

As can be seen, the students reacted very positively not only to the course they were taught (before and after the initiation of the new study plan), but also in particular, within the course, they appreciated everything to do with the problems, direct or inverse, involving modelling and applications.

Conclusions

In Engineering degree courses, and no doubt in several others, there is an invisible dividing line between "basic" and "technological" subjects. Under previous study plans this was even clearer, because the degree courses had a Basic Core and a Technical Core. While nowadays the "explicit" dividing line has vanished, in a way it is as if it still existed. Indeed, in informal conversations with students (and ex-students), they commonly talk about their "urgency" to arrive as quickly as possible to the technical subjects at the end of their degree course. They associate several positive characteristics with these subjects: their evident usefulness, their applicability – in many cases, in the short-term – and many of them also take a favourable view of the minimal time spent on proofs and first principles in these subjects, which instead are absolutely pragmatic. Two examples of this are semi-empirical formulas (which work, although they cannot be rigorously proved), and the kind of tables that suggest what has to be done (for instance, tables of recommended isolation thicknesses, or loss of charge in pipes, etc.), which also provide useful, although not completely rigorous, information.

At the opposite end of the spectrum are the "basic" subjects, particularly the different branches of mathematics. These subjects are where students often find the other side of the coin: rigorous theoretical deduction, coherent and frequently elegant developments of mathematical arguments, strictly consistent notation, etcetera, but at

the same time they also find some less desirable characteristics, such as: dubious applicability, learning concepts that are under-utilized later, a kind of "lack of fit" between a given concept and its practical application, and in general a certain disconnection with technical subjects.

The big challenge, then, appears to be the connection between the expository rigour of basic sciences – especially mathematics – and the practical applicability of technological subjects (Physical Process Engineering, Chemical Process Engineering, Reactor Design, etcetera).

Our own experience (from 1996 to 2002) and other studies published in Latin America [17] seem to suggest that the "bottleneck" in achieving this is the training of human resources. Other studies – from other continents – also pose aspects of the need to train qualified teachers [18-19], which may indicate that the problem is not limited only to this region.

The task is not an easy one, at least, not in Latin America. Training human resources is a process that takes several years, and at the end of the training there is no guarantee that personnel will stay with the institution. However, this instability cannot be used as an excuse for doing nothing. If we are aware that our courses are not well-suited to the needs of degree students and professional life, we cannot remain motionless; on the contrary, we must do everything in our power so that the problems found in real life are also found in our classrooms.

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A Different Perspective for Pre-Integrals and Lyapunov Functions Teaching in Engineering Math Courses: A Real Problem Study

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In courses on Differential Equations, pre-integrals and Lyapunov functions are usually tools for the qualitative study of solutions. In particular, they are commonly used to study stability and in creating phase diagrams. In this article we present a Corrosion Engineering problem in which pre-integrals are a substantial part of the analytical solving method. This allows a completely different, and complementary, approach than the traditional one, with pre-integrals being used as a solving instrument; they also cease to be a simple exercise and are instead directly linked to a real life problem.

Keywords: pre-integrals, Lyapunov functions, corrosion problem.

Introduction

The subject of the stability of solutions in ODE and ODE systems is very important, and many textbooks and specialised web pages devote one or more chapters to it [1-2]. In Uruguay, in particular, special emphasis is given to everything referring to pre-integrals and Lyapunov functions³ [3-4], no doubt because of the importance of Massera's Theorem [5-6]. In fact, in 1949 J. L. Massera proved the converse of Lyapunov's Theorem [5], probably the most important contribution of Uruguayan mathematicians to this discipline [6].

Over and above mentioning the theoretical results, this topic mostly comes up in the practical sheets used for all Differential Equations courses at the Uruguayan public university (UdelaR) [7-9]. At UdelaR, there are only three undergraduate courses on Differential Equations⁴, in the Faculties of Sciences [7], Engineering [8] and Chemistry [9], and all of them include this topic, as can be seen in the following exercises:

Faculty of Sciences

Let $f : [-a, a] \rightarrow \mathbb{R}$ be a continuous and Lipschitz function, such that $f(x) = 0 \Leftrightarrow x = 0$. Prove that the null solution of $x' = f(x)$ is stable if and only if $x \cdot f(x) < 0, \forall x \in [-a, a], x \neq 0$. In this case the solution is asymptotically stable, and $g(x) := \int_0^x -f(t) dt$ is a positive defined Lyapunov function.

³ The term pre-integrals refers to a given function that remains constant over the trajectories of the considered, while a Lyapunov function requires other conditions about the function itself and its derivative.

⁴ Other UdelaR faculties teach courses containing elements of Differential Equations, but do not devote an entire course to them. The situation is similar at private universities.

Faculty of Engineering

Consider $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C^2 function.

Given the equation:
$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = -\frac{\partial H}{\partial x}(x, y) \end{cases}$$

(a) Prove that H is a pre-integral.

(b) Given the equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) \end{cases}$$

Sketch the solutions of the differential equation on the phase diagram, justifying the ideas used.

Faculty of Chemistry

Find the equilibrium solutions to the ODE $\dot{y} = 2y(1 - y)$ and identify them as weakly stable, asymptotically stable or unstable.

While the above is only an example, it is an accurate illustration of the problem sheets, which are basically of three types:

- Exercises that can be solved more or less mechanically and bearing little connection with real life (like the Faculty of Chemistry exercise, above).
- Exercises or theoretical problems that are completely self-contained, whose interest is essentially mathematical, but do not have a direct connection with real-life problems (like the Faculty of Sciences exercise, above).
- Exercises that have some application to real problems (like the Faculty of Engineering problem¹, above) but without that connection being explicit.

As can be seen, the practical exercises and evaluation process involve routine tasks that can hardly increase the intrinsic motivation² of engineering students.

Here we will analyse a real problem in the field of Corrosion Engineering, which has already been solved with other semi-numerical techniques [10-11] as well as analytically [12]. We will now use an approach based on pre-integrals, to solve it and also to provide a practical application that could be used in Differential Equations courses.

A Real Life Problem

In this section, an interesting engineering example will be exposed and pre-integrals and Lyapunov functions will be used to find analytical solutions for the corresponding ODE system. In order to explain these ideas, we will start with the modelisation of a corrosion problem in the tubes of a heat exchanger.

There are different mechanical, heat transfer, reliability and cost requirements for the materials used to construct condensers in power plants, so they are normally made of different materials. A condenser will usually have a shell of carbon steel, tubes made of titanium, stainless steel or copper alloys, and tubesheets of a different copper

¹ The first part of this exercise corresponds to a generalization of the pendulum problem, while the second one is a particular case.

² This comment refers to present-day courses. Previously, other kinds of problems were used (see references [4] and [6]).

alloy.

All these alloys have different resistances to electrochemical corrosion, and since they are in electrical contact they are subject to severe bi-metallic corrosion [13]. In a condenser, high densities of galvanic current are to be expected at the junction of the shell with the tubesheets, and at the junction of tubesheets with tubes, with the current falling off to insignificant values in the tube as distance from the tubesheet increases. If cathodic protection has been provided for the shell, the tubesheet will be cathodically polarised; there will be no effect on the tube, which will behave – or distant parts of it will behave – as an anode. This corrosion can occur in a localised manner, perforating the tube and causing unit breakdown. See Figure 1.

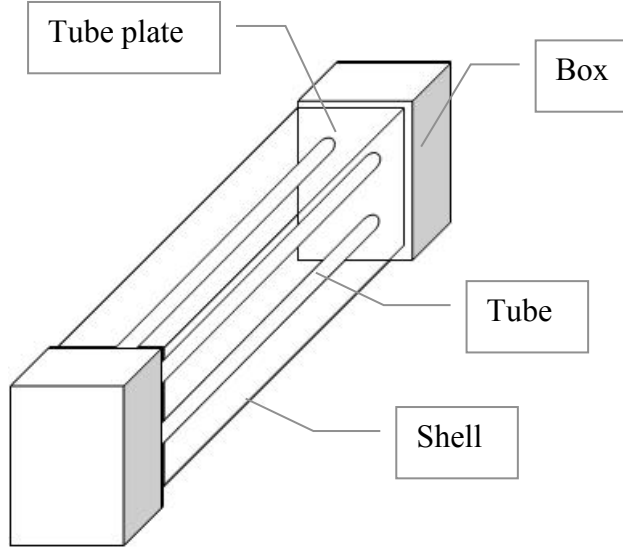


Figure 1. Heat exchanger scheme.

In primary (ohmic drop control) or secondary (charge transfer control) current distributions [14], the transport overpotential is negligible and there is no concentration gradient.

$$\nabla \cdot j = -\chi \nabla^2 E = 0 \quad (1)$$

Hence

$$\nabla^2 E = 0 \quad (2)$$

This equation is formally the same as that used for heat conduction or mass diffusion in a stationary state. The solutions obtained for such phenomena can be applied to solving the electric field in an electrochemical system (with suitable boundary conditions). The solution procedures may be analytical, analogical or numerical.

The system under consideration is a heat exchange tube, so it is convenient to express (2) in cylindrical coordinates

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E}{\partial \theta^2} + \frac{\partial^2 E}{\partial z^2} = 0 \quad (3)$$

For symmetry reasons, we cancel the derivatives with respect to the polar argument θ . The elliptical PDE to be solved is then:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} + \frac{\partial^2 E}{\partial z^2} = 0 \quad (4)$$

Also because of symmetry, we expect zero flow in the centre of the tube, that is

$$\left. \frac{\partial E}{\partial r} \right|_{r=0} = 0 \quad (5)$$

At the tube wall (i.e. for $r=R$), conditions are derived from electrochemical experiments in the laboratory; the result is the relation between circulating current and applied potential (called polarisation curves). As an approximation, two models – linear and semi-logarithmic – of the border condition can be proposed [15]:

$$\frac{\partial E}{\partial r} = K(E - A) \quad (6)$$

Or

$$\frac{\partial E}{\partial r} = B' \exp(K'(E - A')) \quad (7)$$

In the working conditions of the present study, we shall apply the logarithmic boundary condition (7).

Atsley [16] models the problem of current and potential distribution in cylindrical geometry. He outlines geometrical and physical conditions of the system for which a unidirectional approximation in direction of flow is valid, which he expresses as:

$$R < 2[\rho(di/dE)]^{-1} \quad (8)$$

where R [m] and ρ [$\Omega \text{ m}^{-1}$] are the radius of the tube and the resistivity of the electrolyte, respectively, and (di/dE) is the slope of the curve of current density versus potential.

This way, the radial terms are cancelled out. In the conditions outlined, Atsley reports that:

$$\frac{\partial^2 E}{\partial z^2} = \pm \frac{2\rho i}{R} \quad (9)$$

where E [V] is the electrochemical potential dependent on the longitudinal position z , and i is the density of circulating current [A m^{-2}] normal to the tube walls.

In summary, we have the ODE representing the system's potential field (equation 9), the boundary condition at the tube walls arising from the experimental relationship expressed in the polarisation curve, and constant potential is assumed at the ends of the tube:

$$E|_{z=0} = E_1 \quad E|_{z=L} = E_2 \quad (10)$$

An Analytical Solution Using Pre-Integrals

For the ODE system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases} \quad (11)$$

The function $V(x, y) = \frac{1}{2}y^2 + \int_0^x f(t)dt$ is a pre-integral; in fact, over a given trajectory $(x(t), y(t))$ (where t is the independent variable), the following holds true:

$$\dot{V}(t) = \frac{\partial}{\partial t} V(x(t), y(t)) = \frac{\partial V}{\partial x} \dot{x}(t) + \frac{\partial V}{\partial y} \dot{y}(t) = f(x) \dot{x} + y \dot{y} = f(x) y + y[-f(x)] = 0$$

(12)

Hence $V(t)$ remains constant over the ODE trajectories.

In our case, combining (7) and (9), the ODE is

$$E'' = C \exp(\gamma E) \quad (13)$$

$$\text{Where } C = -\frac{2\rho}{R} \exp\left(-\frac{A'}{B'}\right) \quad \text{and } \gamma = \frac{1}{B'}$$

This can be converted in the following system:

$$\begin{cases} E' = y \\ y' = C \exp(\gamma E) \end{cases} \quad (13)$$

And a pre-integral would be:

$$V(x, y) = \frac{1}{2} y^2 - \frac{C}{\gamma} \exp(\gamma E) \quad (14)$$

And since $V(x, y) = c$ over the trajectory, we obtain:

$$V(x, y) = \frac{1}{2} y^2 - \frac{C}{\gamma} \exp(\gamma E) = c \quad (15)$$

Hence:

$$E' = y = \pm \sqrt{\frac{2C}{\gamma} \exp(\gamma E) + k} \quad (16)$$

Where k is a new constant, with positive or negative sign depending on the values of the variables involved.

For $k > 0$

$$\frac{dE}{dz} = \pm \sqrt{k} \sqrt{\alpha \exp(\gamma E) + 1} \quad \text{with } \alpha = \frac{2C}{\gamma k} \quad (17)$$

$$\int \frac{dE}{\sqrt{\alpha \exp(\gamma E) + 1}} = \pm \int \sqrt{k} dz = \pm \sqrt{k} z \quad (18)$$

Using the variable change

$$\alpha \exp(\gamma E) = \sinh^2 t \quad (19)$$

$$\text{We obtain } \pm \sqrt{k} z = \frac{2}{\gamma} \int \frac{1}{\sinh t} dt = \frac{2}{\gamma} \ln \left(\tanh \frac{t}{2} \right) + C^* \quad (20)$$

Simplifying, we get:

$$\tanh \frac{t}{2} = \exp(\varphi z + \psi) \quad (21)$$

Where φ, ψ y C^* are new constants.

$$\text{On the other hand } \tanh \frac{t}{2} = \frac{\sqrt{\sinh^2 t + 1} - 1}{\sinh t} \quad (22)$$

Combining (21) and (22) and making use of the variable change in (19)

$$\exp(\varphi z + \psi) = \frac{\sqrt{\alpha \exp(\gamma E) + 1} - 1}{\sqrt{\alpha \exp(\gamma E)}} \quad (23)$$

After algebraic manipulation we can get:

$$E = -\frac{2}{\gamma} \ln \left[\sqrt{\alpha} \sinh(\varphi z + \psi) \right] \quad (24)$$

For $k < 0$, using the change of variable $\alpha \exp(\gamma E) = \cosh^2 t$ we get

$$E = -\frac{2}{\gamma} \ln \left[\sqrt{\alpha} \sin(\varphi z + \psi) \right] \quad (25)$$

In this case it is $\alpha = \frac{2C}{\gamma|k|}$

This includes the above as a particular case.

$$\text{For } k=0, \text{ we get } E = -\frac{2}{\gamma} \ln(\varphi z + \psi) \quad (26)$$

The above discussion is appropriate, since in the generic system considered here, we cannot predict the sign of $V(x, y)$ since this function is the difference between two positive added quantities, which represent the energy of the system.

Comments and Conclusions

Lyapunov functions as well as pre-integrals are useful tools for the study of stability and/or asymptotic stability in ODE exercises and problems, and ODE systems ones.

From the physical point of view, Lyapunov functions and pre-integrals are basically functions that describe the total energy of a system [17] and in some cases – like the example analysed in section 2 – allow us to obtain analytical solutions for non-linear systems. These two characteristics, especially the latter, are not well-developed in the exercises and problems used in the courses – at least those with the highest student enrolment – taught in Uruguay.

We believe the problem described here fills both needs, as it is an application problem taken from real life. Also, pre-integrals are not just used here as a method for studying qualitative behaviour of solutions, but are an integral part of the solution method.

Moreover, the problem presented here, as well as its undeniable interest for Differential Equations courses – ODE as well as PDE, if the unidirectional approximation for the potential field is not used – has other potential uses, for instance in courses of introductory Calculus (integrals, changes of variables, ODE with separate variables, etcetera) and/or courses in Numerical Methods. For Numerical Methods, it can provide examples for topics like numerical solutions for ODE and PDE and interpolation [11], and it is also suitable for showing semi-numerical treatments [10], which are not common in undergraduate courses.

As a final remark, it is important to mention that it is possible to find solutions by wholly analytical means for the simplified problem, once the Frumkin condition is verified. In fact, it can be solved by tackling a simpler, first-order ODE, and then adapting the resulting solution – by means of a variable change – so as to solve the second-order equation. While this approach is ingenious and gives rise to a plethora of solutions [12], there is no guarantee of obtaining all the possible solutions of the problem. As it was showed in this article, analytical solutions to the ODE can also be found by using pre-integrals, i.e., functions that remain constant within the ODE trajectories. This approach can also produce a plethora of solutions, which correspond to some of the cases analysed in the other analytical procedure [12] but again, it does not guarantee finding all the possible solutions of the problem.

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Being Definitive with Definitions

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It is well known that many students entering university struggle with the role, nature, purpose and use of definitions in mathematics. To those who are thoroughly enculturated into mathematics, it is hard to imagine there is an issue, while to those teaching students it is hard to work out where the difficulty lies. The classic distinction between *concept definition* and *concept image* (Tall & Vinner 1981, Vinner 1983), makes an important start, but there remains the issue of what definitions are for and how they are used. Following various linguists and philosophers (Edwards & Ward 2005) it is vital to distinguish between what have been called *extractive* definitions (usage is described and reported, as in a dictionary) and *stipulative* definitions (specifying required properties, as in mathematics). Students are culturally immersed in and familiar with the former, but in order to succeed in mathematics they have to become used to using the latter. In this paper, task-exercises are offered in order to provoke direct experience of ways of working with students and teachers so that they encounter examples of ‘reasoning solely on the basis of agreed properties’ arising from stipulative definitions.

Method of Enquiry

My preferred approach is always experiential, so I approach the topic of definitions through mathematical tasks. My ‘data’ consists of what the reader finds coming to mind as a result of and while engaging in the tasks (Mason 2002). Commentary and analysis is based on multiple experiences of the same or similar tasks.

Local and Global Definitions

Mathematicians are typically naughty in their use of the word *define*. Sometimes they make global definitions which are supposed to be stable over time and place, when they define a new concept. As both Poincaré (1909/1952) and Whitehead & Russell (1962) suggested, a definition is really a short name for a complex of properties. However sometimes mathematicians make local definitions. Consider the two acts of defining:

Define a function on a domain S and codomain T to be a subset F of $S \times T$ such that each member of S appears exactly once.

Define $f(x) = x^2$.

The first is a global definition, intended to stand permanently (unless explicitly modified), whereas the second is a local definition, a temporary label for a particular function and liable to change at any time. It is perhaps not surprising that mathematical definitions are confusing for students encountering such double usage (Alcock & Simpson 2002, 2011). The mathematician may reply that both fit the sense of definition as a ‘shorthand label for a complex of properties’, but a pedagogically aware mathematician could be perfectly content to use the term *specify* when giving something a local label, because this emphasises that attention is being directed to a

particular instance, as opposed to a general category of objects satisfying a list of required properties. Unfortunately, delving into student experience of definition reveals that this is just the beginning of confusions.

Given a definition of a concept, such as *angle between two lines*, or *ellipse*, how do you determine whether a specified object really is an example? If it ‘looks right’ to learners, if it fits their intuition or their sense of common usage, it is quite likely that it will be accepted as such. This may be sufficient for identifying angles between lines (at least until you try to work with dynamic geometry software), but it is certainly not sufficient for a locus produced by some construction that ‘looks like’ an ellipse. Checking that it actually has the required properties is the move that Tall (1991 p20) calls ‘moving from *describing* to *defining*’, which underpins university mathematics and which could underpin school mathematics as well.

Concept Images and Concept Definitions

Tall & Vinner (1981) developed the notion of *concept image* to refer to the collection of associations and images which come to mind when a technical term is encountered. This is in contrast to the formal, mathematical *concept definition*. The notion of concept image is useful as the beginnings of an explanation for why students’ reasoning does not always make use of *concept definitions*. For example:

Fractionated

Consider the word fraction. What images and associations come to mind? Now bring to articulation, either out loud or on paper (since rehearsing in your head allows for too much in the way of slippage and imprecision!), a definition of fraction.

Comment

Informal definitions might include things like ‘parts of a whole’. Consulting a reasonably reputable source such as Wikipedia reveals a typical ‘dictionary-finesse’ because it refers to fractions as numbers expressed as the ratio of two numbers, begging the question of what is intended by number. Even Wolfram MathWorld defines a fraction as a rational number expressed as a/b with no specification of restrictions on a and b . They also define a rational number as a number that can be expressed as a fraction! These are descriptive, or as will emerge shortly, extractive rather than mathematically formal or stipulative definitions.

According to most formal definitions of fractions, a fraction is an ordered pair of integers, the second (the denominator) being positive (Waismann 1961/2003 p55ff). Anyone imbued with ‘equivalent fractions’ may find themselves thinking in terms of rational numbers rather than simple fractions, perhaps like the authors of the Wikipedia and Wolfram entries, not least because most school textbooks move from fractions to equivalent fractions within one or two pages.

Fractionated continued

Which of the following are fractions? $\frac{3}{4}$, $\frac{4}{3}$, $2\frac{4}{5}$, $\frac{2\pi}{3\pi}$, $\frac{\sqrt{8}}{\sqrt{2}}$, $\frac{-2}{3}$, $\frac{2}{-3}$, $\frac{-2}{-3}$?

Comment

According to the informal but circular definitions, all of these are fractions. According to a formal definition, only the first two and the sixth are fractions. Once equivalence of fractions is included, the domain is no longer fractions but rather

rational numbers (equivalence classes of fractions). So all of the examples shown are rational numbers but only some of them are fractions.

As has been recognised by mathematics teachers for some 3000 years or more, there is nothing like an example to elucidate, but also to challenge and to reveal. Indeed an even more revealing probe is to ask students to construct examples for themselves, including extreme examples and perhaps even non-examples that they think some people might mistake for examples, in order to get them to explore and develop their *personal example space* (Watson & Mason 2005) as part of their *concept image*.

Another and very fruitful approach to fractions is as operators or actions on objects (usually numbers but also on partitioned shapes). Thus fractions are always accompanied by “of ... (some object)”, just as measures are fruitfully always specified by a number together with an error interval. The arithmetic of fractions can then be developed as an arithmetic of operators, with equivalent fractions being actions having the same overall effect on all objects, introduced only when fractions as actions have become familiar.

When introducing a new concept (here ‘fraction’) it is important to remain within that domain until students have become familiar with the concept, that is, having developed a sufficiently complex concept image in order to be able to function with the concept, before increasing the complexity. A similar but different phenomenon surrounds the use of the word *number*.

Number

What is a suitable definition of ‘number’?

How many times in school does the word change its meaning?

Comment

The word ‘number’ is used in schools for whole or counting numbers, for whole numbers with zero, for integers, for rationals (and also for fractions), for some specific irrational numbers such as $\sqrt{2}$, π and e , as well as for reals and complex numbers. It is evident that sometimes students have a concept image come to mind based around counting when the teacher is thinking ‘real’, and so on. It is no wonder then that communication breaks down and learning suffers. At university students meet complex numbers and other number-like constructs such as matrices.

In the spirit of Socrates/Plato, and following the development of axiom systems from Euclid to Peano, concepts in mathematics are defined in terms of specified properties that they must satisfy. This is a sophisticated move as we shall see, and could be supported in school by working with students on mini-axiom systems.

There is considerable evidence (Clarke 2011) that most, even all students in school can undertake sophisticated mathematical reasoning, at least until it comes to the need for calculations in arithmetic or algebra. Few students experience algebra as a language for reasoning, being driven through algebraic calculations to solve equations rather than dwelling in reasoning-with algebra. The same applies to fractions: calculations are ubiquitous and the principle purpose. Reasoning-with is rare.

Extractive and Stipulative Definitions

Edwards & Ward (2004) report surprise that not only do many students not view

definitions the way mathematicians do, but they do not use them appropriately, even in the absence of anything else to do. They build on philosophy and linguistics to distinguish between two types of definitions (Edwards & Ward 2004 p412): extractive definitions describe and report usage, as in a dictionary, whereas stipulative definitions specify usage, as in mathematics. Extractive definitions depend on social enculturation (the terms in the dictionary need to be familiar for the definition to be meaningful) and often build on sensory experience, whereas stipulative definitions are (supposed to be) free from connotations and use only previously defined (or assumed) terms. Other authors have used *intensive* for *extractive* and *extensive* for *stipulative*.

Thus a stipulative definition for 'fraction' might be the formal one, whereas an extractive definition might be to do with 'parts of a whole' or something similar to the ones quoted. An extractive definition of 'negative 1' might involve places to the left of zero on the number line, reference to debts, low temperatures and other instances, while a stipulative definition might be given in the form '-1 is that number which when added to 1 is 0'. Something similar can be used for reciprocal and for logarithms (the log of a number to a given base is the power to which the base must be raised to give the number). These can be treated as slogans which provide a computational base for reasoning-with the concept.

An extractive (or intensive) definition of a continuous function, based on sensory experience, might be 'a function with no gaps or jumps', whereas an extensive definition requires formal apparatus such as ϵ - δ . To move from an intensive sense of continuity to an extensive definition which enables unusual, non-formula-based functions to be analysed for continuity requires fundamental shifts in thinking. Not only is there a change in how to think about continuity, from having a sense of discontinuity to having a way to prove the absence of such discontinuities, but there is a necessary development which extends intuition to include the discontinuity of $\sin(1/x)$ at $x = 0$. Experience of this shift could be available from early on if stipulative definitions were employed more widely, and if reasoning based solely on previously agreed properties was conducted from early on in mathematics lessons.

Extractive definitions are related to concept images, to supporting meaningful construal by students. Stipulative definitions in mathematics comprise the concept definition, and are intended to be used as the sole basis (initially at least) for recognising instances and for reasoning-about and reasoning-with the concept. So when a student is told that $|x|$ is an example of a function whose derivative is discontinuous at 0, the natural response is to reason using concept images rather than resort to checking the specific formal definition. This may seem expedient at school, but is a major stumbling block on entry to university, and it really need not be the case.

A concept such as $\sqrt{2}$ has a stipulative definition as the positive number whose square is 2. An existence proof is required before continuing to use it, though that is very much a mathematical after-thought in the historical development. Similarly for \sqrt{n} where n is a positive non-square integer, and then $\sqrt{-1}$ is stipulated to be a number whose square is -1. Until some existence theorem is proved (by working with ordered pairs of real numbers), a label is required (such as i with the sole property that $i^2 = -1$) so as to avoid confusion between $\sqrt{-1}$ and $-\sqrt{-1}$. Where $\sqrt{2}$ and $\sqrt{-1}$ differ is that unlike $\sqrt{-1}$, several procedures can be specified that give closer and closer rational approximations to $\sqrt{2}$, which therefore can serve as surrogates for recognising and reasoning-with $\sqrt{2}$. The historical challenge for mathematicians, re-experienced as a contemporary challenge for students, is to let go of connotations and the desire for some re-presentation in the familiar, and work solely with stipulated properties. Of

course in the Argand diagram, reinterpreting the Cartesian plane as the field of complex numbers by rotating the real number line through 90° can provide extractive and metaphoric support to intuition.

Students, brought up in a culture of extractive definitions, are called upon at the end of school or early university to switch from treating definitions as extractive to treating them as stipulative and this poses severe challenges. The notion of *concept image* is pedagogically useful precisely because labels for stipulative definitions are usually chosen from familiar vocabulary, thereby carrying extractive and hence imagistic connotations where none were intended. Except of course that the formal stipulative definitions draw upon the imagistic intuitions that are being expressed but displaced by the formal specification.

Thus images of $\sqrt{2}$ as (the length of) the diagonal of a unit square and associations with 45° , Pythagoras, and irrationality contribute to the concept image, but may dominate the two stipulative properties that it is positive, and that its square is 2. For reasoning with and about $\sqrt{2}$, these are all that can be used. The extra shift that $\sqrt{2}$ ‘knows all its decimal places’ (they are all uniquely determined) despite the fact that we can never know them all is but one instance of the conceptual shift to abstraction called upon by the notion of the non-finite. It is highly productive in mathematics to consider infinite processes that unfold iteratively but cannot be completed in material time to nevertheless be completed. Becoming at home with both unfolding and completed infinity is just one of the experiences captured by the pedagogical construct of *procept* introduced by Gray & Tall (1994) as a contribution to understanding, guiding and appreciating the difficulties in reification of experiences, which themselves underpin mathematics.

Extending ‘Number’

For a taste of the process of extending ‘number’ several times in school, consider the following two contexts.

Remainders and Negatives

Finding the remainder on dividing 3^7 by 7 is straight-forward. But what about $(-3)^7$ on dividing by 7?

Remainders of Negative Numbers & Negative Moduli

What would be a suitable definition for the remainder on dividing -17 by 5?

Generalise!

What would be a suitable definition for the remainder on dividing 17 by -5?

Generalise!

What is meant by ‘suitable’?

Comment

Many people consider the possibility that the remainder on dividing -17 by 5 is the same as the remainder on dividing 17 by 5 (appealing to symmetry; replacing something I don’t know how to do with something I do know how to do) and so they get 2. However this does not yield consistency if the arithmetic properties of remainders are extended to apply to negatives as well.

There are several extractive definitions of remainders, based on different approaches to calculating them. A ‘suitable’ definition is one that is consistent with the laws of arithmetic. What properties do remainders and moduli have that one would like to preserve? For example, if you add two numbers and then take the

remainder, you should get the same thing as taking the remainders, adding, and then taking a remainder if that is necessary. Similarly, if you multiply by some number and then take the remainder, you should get the same thing as taking the remainder, multiplying and then taking the remainder if need be.

There is still a choice to be made: should the remainder always be positive (then you can think of arithmetic taking place within that 'set of remainders') or should the remainder lie between 0 and the modulus?

This leads to the question of which properties to maintain: might there be some others that are overlooked but not maintained by the extended definition? Perhaps the most important is that adding a multiple of the modulus to a number leaves the remainder invariant.

Using only the additive and scaling properties, and requiring remainders always to be positive, you can deduce that $(-x) \bmod m$ is the non-negative integer which when added to $x \bmod m$ gives $0 \bmod m$. In other words, $(-x) \bmod m = (m - x) \bmod m$.

GCD

The concept of the LCM and GCD of a pair of whole numbers is familiar. There is a relationship between the GCD and the LCM of a pair of numbers, there are other relationships when three or more numbers are involved, and there are properties of GCD. All of these could be explored by students as a domain in which to reason with a very limited set of properties, while gaining familiarity with those properties. But perhaps more is possible:

GCD of Fractions

What might it mean for a fraction number to be the GCD of two other fractions?
What properties of the GCD of two integers ought to be preserved?

Comment

There may have been a moment of shock ... this is not in the textbooks, and it challenges familiarity with GCD. Yet this is exactly what students experience at various stages in school, and especially on transition to university.

Properties of GCD include: $GCD(\lambda a, \lambda b) = \lambda GCD(a, b)$ and $GCD(ab, c) GCD(a, b) = GCD(a, c) GCD(b, c)$ and $GCD(a + b, b) = GCD(a, b)$.

It would be useful to have the GCD of fractions (as operators) to be invariant when equivalent fractions are used. In other words, the GCD ought to apply consistently to rational numbers.

Take as a stipulative definition that the GCD of n and m is the largest number that divides into both n and m integrally. Now modify this to refer to rationals.

GCD & LCM of Rationals

If the GCD of two rational numbers is the largest rational number to divide into both numbers integrally, is there always such a rational? Is it unique? Is it consistent with the GCD of whole numbers seen as rationals?

What about the LCM?

Comment

If you treat the definition as stipulative, letting go of connotations and association, and if you work solely with the formal stipulative definition, you can find

a definition in terms of the GCDs of numerators and denominators that satisfies all the desired properties.

This task offers teachers another taste of what students might be experiencing many times in school and in early university until they get used to ‘reasoning solely on the basis of agreed properties’. Intuitions (actually, habits with strong concept images) may be strongly challenged when a concept is extended. Other examples might include extending GCD and LCM to numbers of the form $a + b\sqrt{n}$ where a and b are integers and n is a fixed integer which is not a square. Exploring this would give practice in manipulating surds.

Tertiary Examples

Consider the terms *increasing* and *decreasing* in the context of infinite sequences. Alcock & Simpson (2011) and Edwards (2011) have explored students’ grasp of these definitions, which sometimes conflict with natural language usage. Take as a working definition that an increasing sequence is a sequence $\{u_n: n = 1, 2, \dots\}$ for which $\forall n, u_n \leq u_{n+1}$.

Increasing & sequences

Construct some sequences that meet the definition but which might be mis-classified by students.

Comment

Alcock & Simpson (2010) and Edwards (2011) show that students use different interpretations of the concept in different situations. Yet mathematicians know that definition-checking involves strict application of the details of the definition. In other words, definition-checking involves checking that required properties are correctly instantiated in the particular situation. However, it is very easy to overlook what turn out to be essential ingredients. Thus even having read the definition once or twice, it is easy to overlook the inclusive inequality condition and construe it as as expressing intuition (strict inequality over the long run) without attending to the implications of the equality or the requirement that it apply to every consecutive pair of terms.

The issue may boil down to whether student attention is suitably drawn to the features that matter: mathematically, ‘increasing’ has to be everywhere not just in some places or as an overall trend. To ensure that student attention is suitable directed takes more than loud voice tones and deliberate pointing on a screen. Student and tutor attention alike need to be aligned so that both are attending in the same way (Mason 2004/2008).

Even ‘knowing’ definitions and characteristics of something is not sufficient for generating mathematical reasoning, as Peled & HersHKovitz (1999) found amongst pre-service teachers working with irrational numbers. What appeared to be missing was flexibility in shifting between (re)presentations to support thinking in different contexts. So students are expected on the one hand to use stipulative definitions rigorously (not doing so was what seemed to lead Alcock & Simpson’s students astray), and yet on the other hand to be familiar with and to use alternative characterisations as accessed in different (re)presentations (as Peled & HersHKovitz’s teachers did not). No wonder first year undergraduates get confused!

Root-slopes and inter-rootal distances

The notion of inter-rootal distances for a polynomial is not very familiar, nor is

the notion of the root-slopes taken to be of particular significance. But both are closely allied to the generalisation of the discriminant of a quadratic. These concepts provide an opportunity to experience what it may be like for students to encounter new notions and be expected to work with them straight away.

Root-Slopes & Inter-Rootal Distances for Quadratics

What is the product of the root-slopes (the slopes at the roots) for a quadratic such as $x^2 - 4$? What about in general?

What is the inter-rootal distance (distance between the roots) for a quadratic?

Comment

Squaring the inter-rootal distance suggests that it is closely linked to the discriminant even when the roots are complex. The square of the inter-rootal distance is similarly closely linked to the discriminant.

Given a polynomial function f , consider the square of the product of the inter-rootal distances. If $f(x) = a_n \prod_{k=1}^n (x - r_k)$ then the product of the inter-rootal distances is

$\prod_{j=1}^{n-1} \prod_{k=j+1}^n (r_j - r_k)$. The square of the product of the inter-rootal distances is then

$\prod_{j=1}^n \prod_{k \neq j} (r_j - r_k)^2$ while the product of the root-slopes turns out to be

$a_n^n (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n-1} \prod_{k=j+1}^n (r_j - r_k)^2$. Since the discriminant is usually taken to be

$a_n^{2n-2} \prod_{j=1}^{n-1} \prod_{k=j+1}^n (r_j - r_k)^2$, the three results agree to within a power of the leading

coefficient and a power of -1.

Using the theme of *doing & undoing*, suppose someone gave you a list of numbers and claimed that these were the root-slopes of a polynomial. How could you check that assertion? In other words, what properties must the set of root slopes have? It turns out that for a polynomial of degree d , the sum of the products of the root-slopes taken $d - 1$ at a time must be 0, and this is a sufficient condition for cubics but not quartics or higher.

Tangents

Student difficulties with the concept of tangent are legion (for example, Vinner 1982, Tall 1986, Winicki-Landman & Leikin 2000, Biza 2011), and inflection points are equally complex for students (Tsamir & Ovodenko 2004). There is often a conflict in concept image between ‘touching the curve in one point’ and ‘touching the curve in one point at the point of tangency’. Here is a task that may provoke learners to interrogate their sense, their concept image of what a tangent is while at the same time ‘discovering’ inflection points (and their tangents).

Define the *tangent power* of a point P in the plane with respect to a curve C to be the number of straight lines through P which are tangent to C somewhere.

Tangent Powers

Since every point in the plane is assigned a non-negative integer with respect to a

given curve, the plane is divided into regions with constant tangent power. For the quintic shown below, sketch the regions of points which have the same tangent power. Make some conjectures about the quintic, and about functions in general.

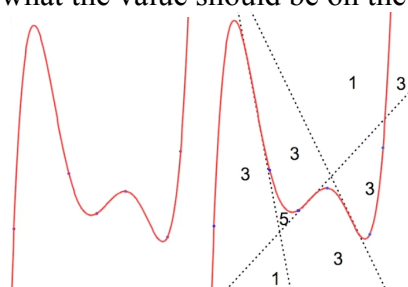
Comment

There are at least two ways to explore this question. One is to select a point P and then rotate a line through P about P seeing how often it is tangent. This may challenge the appreciation of what happens as x gets very large in absolute value, but it proves tedious to do for a lot of points. An alternative is to imagine a straight line tangent to the curve at point T (a pen or pencil is handy for this) and then run T along the curve, getting a sense of what the tangent does, and then articulating the boundaries of regions where the tangent changes direction.

In this way, learners encounter the second derivative (where the tangent changes direction), and because they are attending to the point T , they tend not to worry about whether the line crosses the graph, or intersects it somewhere else as well.

The shift of attention from lines through a point to points encountered by a shifting tangent is typical of mathematical shifts, where the property that is wanted can be experienced in two (or more) ways, and what is important is flexibility between these.

Various conjectures may arise about how the tangent power changes as you cross from region to region, and what the value should be on the boundaries of regions.



The theme of *doing & undoing* suggests the question

Given a finite family of straight lines in the plane is there a polynomial (or a function that is at least twice differentiable?) for which the given lines are the inflection tangents for the function? If not, what condition(s) must they satisfy?

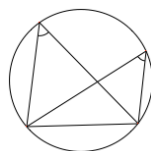
Notice how the shift of attention from number of tangents through a point, to the number of times a moving tangent passes through the point opens up possibilities. It actually offers more properties to work with.

Contexts for Work Prior to University

This section contains brief indications of topics in school mathematics that could be used to provide experience of the mathematical use of stipulative definitions.

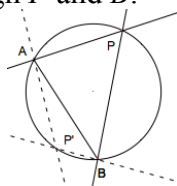
Subtended Angle

There is a well known Euclidean theorem that the angles subtended at a circle on the same side of a chord are equal.



One irritating feature of this result is the condition that the point must always be on the same side of the chord as the centre of the circle. It is awkward and inelegant to have to keep repeating that phrase, and to have to keep checking it when using the result in some context. Furthermore it overlooks the fact that angles on opposite sides are supplementary. It would be so much more elegant and simpler to be able to assert that the angle subtended at the circumference of a circle by a chord is invariant, but this would have to be balanced by the statement of the theorem that the opposite angles in a cyclic quadrilateral are supplementary.

Examination of the diagram shows that the ‘angle subtended by a chord’ is really highly ambiguous, because it specifies an angle between two straight lines: the line through P and A , and the line through P and B .



Which angle should be chosen, or how might one of them be specified so as to make the theorem work for all points on the circumference?

Imre Lakatos (1976) is well known for his radical and ground-breaking book *Proofs & Refutations*, in which he uses the historical development of two mathematical theorems, one in geometry and one in analysis, to propose that definitions and theorems co-emerge as mathematicians struggle to locate hidden assumptions which permit unintended counter-examples. What exactly then, is ‘the angle between two lines’? Is there a unique such object?

Angle Between two Lines

What is the angle between two lines meeting at a point P ?

Comment

One approach is to declare the smaller of the two angles to be the angle, so that the angle between two lines is always less than, or equal to, 90° . This causes dynamic geometry programs all sorts of trouble, and it treats angle as unsigned, so orientation is ignored. What are the implications of this choice for the subtended-angle theorem?

A stipulative rather than an extractive definition is needed in order to reason mathematically.

Another approach is to observe that angle measure involves one arm turning towards the other. Define the angle between lines m and n to be the angle - less than 360° - through which m must turn about the intersection point in order to coincide with n . One implication of this approach is that *the* angle between two lines depends on the order in which the lines are specified. What are the implications of this choice for the subtended-angle theorem?

The subtended angle theorem offers an opportunity to clarify and critique different definitions, and to consider implications of the available choices. Elegance is achieved through simplicity, but it may depend on where you want that simplicity to be based!

Modulus

The function $|x|$ (absolute value) is usually introduced as if it were being defined extractively: some sort of description is given to do with removing the ‘sign’ or taking the distance to the origin (Vinner 1991, Leiken & Winicki-Landman 2000). Stipulatively it is an early example of a function specified by gluing together two formulae, as in

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Pedagogically, an extractive definition makes sense because it offers images that can be accessed as part of the concept image, but for calculation purposes, the stipulative is essential. Apart from the classic tasks of sketching the graphs of the relations $\{(x, y): |x| + |y| = 1\}$ and $\{(x, y): |x + y| + |x - y| = 1\}$, there is plenty to explore by thinking of $| \cdot |$ as an action performed on other functions. Thus the functions

$$f(x, a_1) = |x - a_1|, f(f(x, a_1), a_2), \text{ etc.}$$

provide a succession of images (zigzags) to explore, as well as a context for developing familiarity with how the absolute value function works. Using the doing-undoing theme gives rise to trying to classify those zig-zag graphs that can be obtained by such iterations. Multiplicative factors could also be introduced in order to alter the slopes.

Using a construct (absolute value) as the basis for exploration gives rise to mathematical exploration through which learners can experience various aspects of mathematical thinking (Mason, Burton & Stacey 1982/2010). It also provides another experience of reasoning solely from and with a few agreed properties as a taste of the role and importance of mathematical definitions and axioms.

Squares & circles

Encountering different definitions for the same thing can be very confusing. For example, $\sin(x)$ or $\sin(A)$ is defined variously as a ratio of sides of a right-angled triangle, a function generated by a point moving round a circle, the solution to a differential equation, and a power series. Many mathematical theorems take the form “the following properties are equivalent”, so that any one of them can be taken as a definition (see for example Winicki-Landman & Leiken 2000 and Leiken & Winicki-Landman 2000).

What then is a square? The English verb ‘is’ is highly ambiguous concerning whether what is intended is a list of properties, or a parsimonious stipulation. From a property point of view, a square is a quadrilateral with four equal sides, four right-angles, opposite sides parallel, equal diagonals that bisect each other at right angles, a perimeter of four times the length of one side, an area of the square of the length of a side and an area which is the square of one quarter of the perimeter; it is also a shape that is involved in theorems such as Pythagoras, and which gives its name to the algebraic expression x^2 . One could go on, listing, for example, other shapes of which it is a special case, such as rhombuses, parallelograms, quadrilaterals, and other properties.

It could be valuable for learners to become aware of the fact that shapes have many properties. However, it is an obstacle if they come to believe that you need to check all of those properties before you are sure that you have a square in any particular situation! Of course that seems ridiculous when confronted with this long list of properties, but it applies equally to the restricted properties of having four equal

sides and four right angles, and having equal diagonals that intersect at right angles.

If you want to be able to use the properties of squares for making deductions, you need to have a reasonably parsimonious but psychologically sensible collection of properties that guarantee all the others⁵. This is the mathematically ubiquitous notion of *necessary and sufficient*. In the case of a square, what choices of necessary conditions are sufficient to guarantee that a shape is indeed a square? A similar exercise can be carried out with circles, rectangles, kites, as well as with numbers and other school constructs. Finding a necessary and sufficient set of properties is precisely the approach advocated by Plato through Socrates, when switching from searching for an extractive definition of abstract concepts such as beauty or truth, to a stipulatively listing necessary and sufficient properties.

What can be engaging for learners is to try to find counter examples to proposals they and others make as to necessary and sufficient ‘minimal’ conditions. For example, looking for a shape that has one right angle, one pair of opposite sides equal and one pair of opposite sides parallel, but which is not a square can involve a considerable degree of geometric thinking, which may be more valuable than memorising lists of properties.

Magic square reasoning

Definition: a magic square is a square array of numbers for which the sum of all the entries in any row, in any column, and on the two main diagonals, is the same. Their properties provide a mini-set of axioms from which numerous deductions can be made, as in the following.

3 by 3 Magic Squares

Why, for any 3 by 3 magic square, must the sum of the red (dark shaded) squares be the same as the sum of the blue (light shaded) squares in each diagram?



What is the scope of ‘any’ in ‘any 3 by 3 magic square’?

What other configurations can you find for which the sum of the dark/red shaded squares is the sum of the light/blue shaded ones? Do all colour-symmetric configurations work (interchanging colours is a geometric symmetry)?

Comment

The only properties available are the requirements that the row, column and diagonal sums are the same. So these properties act as a mini-set of axioms for reasoning-with. Because no actual numbers are available, an empirical approach is stymied. Consequently, learner attention is directed to manipulating the properties available using but one idea: if the same term appears in two equal sums, then it can be eliminated from both without changing the fact of equality.

⁵ Matthew Inglis (personal communication) pointed out that there is often a trade-off between efficiency and assimilability, between parsimony and intuition.

4 by 4 Magic Squares

Why must the sum of the dark/red shaded squares be the same as the sum of the light/blue shaded squares in each diagram?



What other configurations can you find for which the sum of the dark/red shaded squares is the sum of the light/blue shaded ones?

Creating your own configurations is relatively straight-forward; justifying someone else's configuration can be much more challenging, and this is typical of the doing-undoing theme in mathematics. The point of these and other tasks is that reasoning-with the few known facts (axioms or properties) not only offers opportunities for exploration and creativity, but provides useful experience of reasoning solely on the basis of agreed properties, that is of using stipulative definitions mathematically. Calling upon this when appropriate in standard curriculum topics would enrich student experience and prepare them for university mathematics.

What Can Teachers Do?

Preparing students for university mathematics calls for more than repetition of procedures so that they become second nature. It also involves exposing students to the mathematical use of definitions. Teachers in schools can:

- Be aware of different ways of attending to objects: in terms of informal 'extractive definitions' based on images and metaphors, and in terms of 'stipulative definitions' based on necessary and sufficient properties;
- Be awake to similarities and differences between natural language usage and technical use such as *define* and *specify*, and to *extractive* and *stipulative* definitions when they appear;
- Develop adequate concept-images through the use of examples where the pertinent properties of examples are highlighted;
- Prompt students to construct their own examples, including 'boundary examples' and near-miss examples;
- Use mini-axiom systems with students in order to provide experience of movement from recognising relationships to perceiving properties.

Definitions are chosen so as to be *definite* about the terms we use, but also to make both the statement of theorems, and the reasoning required to justify them, as elegantly simple and hence as memorable as possible. Engaging learners in the implications of different variations of a definition may lead to an appreciation for learners of the role of definitions, and the reasoning that those definitions enable.

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Mathematics in Science Higher Education: Narrative Inquiry and an Analytical Framework for Exploring the Student Experience

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Efforts are emerging to link the teaching of mathematics and science in higher education in response to a myriad of reports on the declining mathematical abilities of science students within an environment where science is becoming increasingly more quantitative. However, research in the area of interdisciplinary mathematics-science education at the tertiary level is lacking, as are research methodologies and theories to guide those exploring this emerging territory. This paper aims to make a contribution in this space, offering a qualitative methodological approach with a theory-informed analytic framework. Drawing on research from a three-year longitudinal study, narrative inquiry is employed to present the complexity and richness of the holistic student experience to more deeply understand how beliefs of mathematics are influenced by the science curricula. A single, in-depth case study is analysed within a framework, inspired by the work of Barnett and Coates, and Barton, to connect how mathematical knowledge, and its application in science can influence students' perceptions of mathematics. Implications for educational research that spans science and mathematics in higher education are discussed, along with implications for practice within a changing tertiary policy environment.

Keywords: mathematics; science; quantitative skills; interdisciplinary; higher education; curriculum

Introduction

There is widespread agreement that Quantitative Skills (QS) are essential for graduate competence and preparedness in science [1-5]. Not only are secondary school students holding negative views of quantitative subjects, they are also underperforming in mathematics and science [6]. QS refer to the application of mathematical and statistical thinking and reasoning within a given external context, science in this case. QS can be thought of in terms of numeracy (as defined by the Australian Association of Mathematics Teachers) or mathematical literacy (as defined by the Organisation for Economic Co-operation and Development), as both have an applied meaning with an eye to connecting mathematics to life beyond the classroom. For the purposes of this paper, QS assume mathematical knowledge and refer to the act of applying that mathematical knowledge in the context of another discipline. In addition, an inclusive, broad view of mathematics is being used in the paper, which incorporates statistics.

The dynamic nature of modern science requires responsive and equally dynamic curricular models to ensure that science graduates are prepared for the needs of the larger world of science. However, evidence suggests that science programs have typically remained static, emphasising content at the expense of students learning the skills needed to apply content knowledge [1,7-10]. Amongst the myriad of calls for action and associated funding opportunities, some institutions are reforming

undergraduate science curriculum with the aim of building QS, such that graduates can apply mathematics to describe and represent scientific phenomena, and to solve scientific problems. As new approaches are implemented to achieve the desired increased QS learning outcomes, research in this interdisciplinary area in higher education is its infancy and fragmented as a result. The majority of research explores single units of study as opposed to mathematics and QS in the larger science degree program, and is often at the level of the educator implementing new unit-specific pedagogies and assessment. For evidence of this, see the special edition of CBE Life Sciences Education (2010: 9[3]), which is dedicated to empirical research on innovations to build the QS of science students.

Centred around the broader question, *how are students' beliefs about mathematics influenced by the undergraduate science curriculum?*, the paper builds on and links the works of Barnett and Coates [11], and Barton [12]. The aims and design of the study are elaborated on below, highlighting the qualitative approaches undertaken to gather data from undergraduate science students prior to completion of their degree. The developed analytical framework incorporates Barnett's notion of 'knowing', 'acting' and 'being' along with Barton's VPOR (vision, philosophy, orientation, role) model, and is interpreted through a 'curriculum as experienced by students' lens. Finally, implications of the analytical framework and implications for further research are discussed.

Considering the student in higher education curriculum

Barnett and Coates [11] described higher education curriculum in terms of the *complex interplay* of 'knowing', 'acting', and 'being'. Based on experience, insider knowledge, relevant literature and interviews with academics, they developed a model to spark conversation about curriculum, an area they felt was neglected in higher education. 'Knowing' refers to the active, dynamic process of gaining knowledge. 'Acting' is the process of applying the knowledge, of putting the 'knowing' into action, deploying skills. Without 'knowing', students are in danger of 'acting' inappropriately. 'Being' refers largely to the individuality of the student, which Barnett and Coates [11] describe using words/phases including: self; capabilities; self-confidence; inner lives of students; and self-reliance. Thus, 'acting' occurs when students practise and engage with the 'knowing', and it is through 'acting' that they transition from *doing to becoming* (or 'being'). 'Being' is viewed as the adhesive that makes 'knowing' and 'acting' meaningful to the student, and to neglect it is to have students who repeat, parrot and mimic. However, 'being' is a vague notion, difficult to quantify and categorise, and it was not the intention of Barnett and Coates [11] to attempt this. In the area of mathematics education research, there are models and theories that lend themselves to further understanding Barnett and Coates' [11] 'being'.

Barton [12] proposed 'VPRO' as a heuristic model for considering the preparation of mathematics teachers: their 'vision' and 'philosophy' of the discipline along with their notion of the 'role' of mathematics in the world, and their 'orientation' for approaching the discipline. Whilst largely considering theories around Mathematical Knowledge for Teaching (MKfT) in primary and secondary school environments, Barton is contributing a model for understanding a teacher's "relationship with mathematics" that extends beyond the simple notion of an educator as person who teaches mathematical knowledge to students. The sense of self and being as an individual, as an educator and as a mathematician, and how this then influences the act of teaching lends itself to Barnett and Coates' notion of 'being' which connects

knowing and the applying of knowledge to a sense of self. Thus, to further explore students' notion of 'being' as discussed by Barnett and Coates [11], Barton's [12] model has been adapted for this study from teacher-centred perspective of VPRO into 'BRO' with the unit of analysis being the tertiary student. Indeed, Barton [12] suggests that students need to be cognisant of their VPRO as well. The model is adapted as the focus is on science students and how they apply mathematics in the context of science, as such VPRO was modified for an applied focus, to extent further than the of learning mathematical knowledge. BRO is the overlapping components of a student's beliefs (B) about their level of mathematics, their sense of the role (R) of mathematics in science, and their orientation (O) for learning mathematics in the context of science.

Conceptualising the student in higher education curriculum reform

The (primary) intended beneficiaries of any curricula reform effort are students and this study is deliberately designed to offer the 'curriculum as experienced by students' perspective. The notion of curriculum is complex, as evidenced by the use of various theories in the literature. Curriculum is considered in light of the model of the planned, the enacted, and the experienced curriculum [13] with a diagram to visualise curriculum as conceptualised for this study displayed in **Figure 1**. More specifically, curriculum incorporates all undergraduate units/subjects with a holistic focus on how students experience the curriculum in relation to the interdisciplinary connections of science and mathematics as articulated via QS.

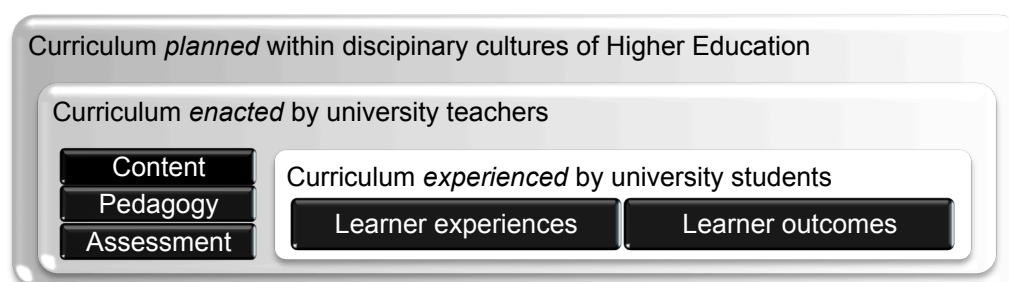


Figure 1: Conceptualisation of curriculum.

Purpose of Study

This paper is situated within a larger, mixed methods research study documenting how a 'QS-intended' curriculum, that is, a curriculum designed with the explicit goal of building QS, influences student attitudes towards, and learning of, mathematics in the context of science. Within the context of the interdisciplinary science-mathematics curriculum and the curriculum as experienced by students, this paper utilises a narrative inquiry approach to present the student perspective of learning and applying mathematics in a science degree program. The story is analysed within a theory-informed framework as a mechanism to further research in the emerging area of interdisciplinary science and mathematics education at the tertiary level.

Methodology

The educational context was a research-intensive university in Australia, with approximately 40,000 students. The Bachelor of Science (BSc) is a large, generalist degree program with more than 3,000 undergraduate students. Applicants are required to have completed high school level English and Mathematics along with either

chemistry or physics. The university was selected primarily because it was in the process of implementing a ‘QS-intended’ curriculum, following an extensive institutional review of the Science program. The study was granted ethical clearance through the University’s Behavioural and Social Science Ethical Review Committee.

Study design

A longitudinal study was employed with data collection occurring at four time points during the three-year undergraduate science degree program. Given the high attrition rate from the science program following first year, participants were not invited until the beginning of second year. Data were collected through a mix of face-to-face interviews, and asynchronous online open-ended questionnaires. **Table 1** offers further detail on the interview time points, data collection method utilised and the focus of the data collection. All face-to-face interviews were audio recorded and transcribed verbatim.

Table 1: Longitudinal Data Collection Method and Focus for Data Collection at the Four Time Points

	Time	Data collection	Topic
T1	Teaching week 5, semester 1 of 2 nd year	Face to face, 1-to-1 interviews	Transition into university, and into 2 nd year, QS interactions in 1 st year and subsequent beliefs
T2	Exam period, semester 2 of 2 nd year	Asynchronous interviews via SurveyMonkey	QS interactions in 2 nd year and subsequent beliefs
T3	Teaching week 5/6, semester 1 of 3 rd year	Asynchronous interviews via SurveyMonkey	Transition into 3 rd year, QS interactions and subsequent beliefs
T4	Teaching week 12, semester 2 of 3 rd year	Face to face, 1-to-1 interviews	QS interactions in 3 rd year, overall reflections and subsequent beliefs

Analysis - narrative inquiry and the analytic framework

The principles of narrative inquiry were employed to present the case study of one student, allowing for a richness of individual experiences to be described fully [14]. The case study was drafted in a narrative format and structured sequentially. The longitudinal data sources for the student were treated as a single source of data, compiled and read in order of first to final year. Data were not analysed or considered as isolated fragments in time but as a series of sequential events occurring over a three-year time period. No framework was utilised in the first iteration of coding, simply a process of reading all the data to establish a sense of the student and their story. Transcripts were simultaneously read with audio playing to document the student’s tone, expressions and verbal aspects that could not be captured by text alone. Then the written data with notes from the audio were re-read iteratively with text highlighted and chunked around the themes of the analytical framework. Keeping in the spirit of narrative inquiry, the coding process did not involve dividing, counting or separating the data and instead focussed on understanding the student’s holistic experience to better represent it as a narrative. Following the iterative process of reading, listening, and then reading and coding repeatedly, the researcher began drafting the narrative. A process of drafting the narrative and then referring back to

the text was repeated over a period of weeks to ensure the story was representative of the student's experience and not assumptions or interpretation of the researcher.

The narrative is presented solely based on the data from the interviews and the student's academic records, and is intended to be told without analysis. The analytical framework applied during the coding phase was for use in the discussion of analysis and to ensure 'fit' of the framework as an appropriate and comprehensive device for interpretation following the presentation of the narrative. The analytical framework synthesizes models from two unrelated studies. Firstly, it builds on the work of Barnett and Coates [11], adapting their model for conceptualising the curriculum in higher education around 'acting, knowing and being'. Secondly, in order to operationalise some of the more abstract aspects of Barnett and Coates' notion of 'being', Barton's [12] VPRO model for mathematics teachers was modified into BRO (beliefs-role-orientation), which was previously described. The discussion section is interpreted in light of the analytical framework.

Study participant

The research is exploring science student learning outcomes and beliefs at a single university. Therefore, the sampling strategy was aimed at reducing influences from outside the institution's science curriculum. Thus, only students in a single science degree program (as opposed to a dual degree) with full-time enrolment and little or no transfer credit were selected to participate in the study. In early 2009, students were invited via email to express interest in becoming a study participant. Fourteen students were selected to achieve a mix of gender, age, and field of study/major. Thirteen students completed the study with one dropping out of the university for personal/family reasons.

The full story of one is presented as an example of how the analytical framework was used in regards to the narrative inquiry approach [14]. The student was selected for heuristic value in that her change in beliefs ran counter to the intentions of the 'QS-intended' curriculum and hence has some implications for improving the teaching of mathematics and the application of mathematics in science at a program level.

Results

The case study is presented as a narrative, following the structure of a story as opposed to data results typical of an empirical study. The story begins broadly with background context, and then follows the student through her three years of study.

Starting a science degree and perceptions of mathematics in science

Midas returned to study science at 28 years of age. She had previously studied science in university, thinking about a future in medicine but switched into an economics degree program, where she graduated and entered the workforce as a tax officer in a government department. After four years of employment, she was not satisfied and returned to university to study undergraduate science with the hope of establishing a more rewarding career in scientific research in the area of biomedicine. She entered her second university degree with a sense of identity as a 'maths person' although she was concerned that her mathematical knowledge would be inadequate for the requirements of science. In fact, she felt disadvantaged from the start because she had 10 years between her high school mathematics subjects and her science degree program.

Three years in science: Building mathematical knowledge and QS in science

In her first year, Midas selected the recommended units for life sciences-focussed students including two biology subjects, two chemistry subjects, two quantitative subjects taught largely by the mathematics department including statistics and a mathematics-science integrated subject, and a physics subject for life sciences students. With her one elective space in first year she selected a mathematics subject because she wanted to “*brush up*” as she believed that mathematics at a sophisticated level would be need for science. In addition, she enjoyed the rules, structure and the methods to mathematics, describing it as “*relaxing*”. Midas expressed a preference in first year for building a stronger mathematical knowledge base, as opposed to applying that knowledge base in her science units. When it came to QS, she stated...

...just teaching maths as a course (subject/unit) would be really good. Because quantitative skills, it's hard to say you need to know how to do these particular procedures. You need to know how to manipulate numbers generally. You just, you need to have an understanding of the how, powers and logs and all that sort of thing work. And you get that from general maths, just I find, rather than when they just sort of tack it on in the lectures in science. Manipulating equations and things like that, I think a lot of people struggle with cos they just sort of show you in the lecture, they don't explain why and how. I think it would make it easier if we did more maths.

She went on to say that this should occur in the university science program, because “*maths is like a language that you need to keep doing to remember. So even though I do remember it a bit more now, it's still not as if I'd just come out of year 12 doing it.*” She suggested that first year statistics and the integrated science-mathematics course should be extended to a full year to allow for more time. Midas' focus in first year was on mathematics knowledge with little attention to applying the mathematics in her science units. Although she completed and enjoyed the integrated science-mathematics course, she never once made mention of the links between mathematics and science. She found the first year statistics course, largely conceptual, to be “*annoying*” as a way of learning, preferring a calculation-based approach to learning statistics.

By second year, Midas was taking all science-based units taught from science departments, which were largely content focused at the expense of buildings skills, including QS. She identified low levels of mathematics being applied, including some solution concentrations (chemistry) and some very basic statistical tests (biology) focussed on getting a p-value. No new mathematics were taught and no exciting applications of mathematics were highlighted in her science-based units. When she discussed QS in her second year, she said there...

...was really no test of these skills at all this year, which is fine if this is a true reflection of the necessary skills in the roles of that my subjects lead to, however I feel there could be more lab based calculations used – calculating volumes, etc, we will actually be using in practicals. I completed a (undergraduate research placement) project over the mid-year break and needed to be able to do these sorts of calculations frequently which was a struggle to start with and it would have been good to have more practice prior to the project. Having said that though I was able to pick it up reasonably quickly so maybe these are skills that can be learned ‘on the job’.

Midas recognised that she had not enrolled in any quantitative-based units, so she had no expected a high level of QS being required. The use of mathematics in second year was “*applied and always in the context of qualitative scenario rather than maths for the sake of maths*”. However, she did lament that she did not have room to select

another mathematics unit in second year. She expressed a desire for curriculum coherence across the year levels, and wanted to see more formal structure for building a mathematics knowledge base that progressed from year to year.

Third year was largely reminiscent of second year in terms of Midas' QS experiences. The place of applying mathematics was in the practical component of her subjects, although she cites "*small amounts of statistical analysis, basic calculations*". By graduation, she was ready to be done with her undergraduate studies, saying: "*It's done what it needed to do, which is give me the bit of paper which I can now go away and use and actually learn stuff.*" She had deduced in second and third year that QS in her area of science, biomedical research, require minimal sophistication in mathematics.

I did a maths course and a stats course in first year. And since then I haven't done any really QS focused subjects. We've done a few statistical tests in a few subjects, but that's pretty much it. A few required calculations of concentrations and that kind of thing, but it's very, very minimal. And I wouldn't say your QS is developed all that well in the science degree.

By third year she believed mathematics could be taught within the context of the science units because she noticed that the level of QS needed was quite basic. In addition, being a mathematically oriented person, she felt confident with the skills she brought from secondary school and her economics degree that she could work out any quantitative problems.

QS just hasn't been that much of an emphasis in the degree. So I think it really is something that you develop more in secondary school and you either have an aptitude for it or not. And I feel that I do, so I'm fine, but it's not as a result of what I've been taught (in the science curriculum).

Midas graduated with a Bachelor of Science, majoring in microbiology in 2010 and started in honours the following year. She plans to enrol in a PhD following her honours year and hopes to build a career in biomedical research as a scientist and possibly become a university academic.

Discussion: Analysis of Case Study Using Analytical Framework

Acting and knowing: QS as the overlap of knowing mathematics and applying it in the context of science

Quantitative skills, QS, by definition relate 'acting' and 'knowing' in terms of mathematical knowledge being applied, taking some action for the student, in a scientific context. Thus Barnett and Coates' model seems apt for the research and theory development in the area of interdisciplinary science and mathematics via QS. Considering Midas' story in this context, it becomes apparent that Midas brings a strong notion of mathematics as a structured discipline with its own method, and this notion of mathematics is appeals to her. She grasps a place for mathematical knowledge in science and clearly pursues development in her first year. Whilst she enters her science degree with a belief that she will need mathematics knowledge for science, she does seem to experience any genuine application of her mathematical knowledge in her science units. As she progresses, her curricular experiences of QS decline and her sense of engaging with mathematics as enjoyable becomes engagement in applying basic mathematics as a tool of science. Barnett and Coates [11] warned that curriculum designed and implementation without space for connecting knowledge and the engagement of acting on knowledge leads to 'skill

acquisition' that is divorced from appropriateness of context, and makes no contribution to the development of the student as a person. This is evidenced as Midas wants to become a biomedical research scientist, which the profession claims is inherently quantitative and mathematical, and yet her entering enthusiasm for mathematics is diminished as she progresses in her studies.

Barton [12] discusses MKfT, stating, "they must teach mathematics not because this bit of knowledge is important, but because these experiences will allow a student to build future mathematical concepts." From Midas' perspective one gains a picture of her experienced curriculum as one where the application of mathematics was not taught as a means of furthering mathematical or scientific concepts, it was simply about simple mathematics as a tool to further science content knowledge. This points to a well-documented weakness in undergraduate science education where content-rich units delivered in passive modes to masses of students is the dominant and default pedagogical approach. In this model, 'knowing' is the overwhelming conception driving curriculum. Indeed, Barnett and Coates' [11] research into science higher education supports just this model of science curriculum, where 'knowing' dominates pedagogy with some, largely disconnected opportunities for 'acting' in practical classes and minimal attention paid to 'being'.

Being: Understanding beliefs, roles and orientation (BRO) for building QS in science

Midas came into the program with *beliefs* about herself as a mathematics person and she graduated with that belief, however she did not attribute the development of her QS to having completed the science degree. In fact, she realised that QS were not something to be learned, they were more an attribute of who you are. Her view of the importance of the *role* of mathematics in science declined as she progressed through the program, from being regarded as essential to eventually being seen as something you could just grab from the 'tool-kit' when needed, which was not that often. Her incoming notion of '*mathematics as the language of science*' was not realised during her undergraduate studies in science. Midas' *orientation* to learning QS changed dramatically from the time she entered her science studies to her graduation. She began the program wanting to learn mathematics to build her mathematical knowledge base, and because she enjoyed mathematics. She wanted more time to learn it and at regular intervals. She commented that first year statistics was '*annoying*' as it was taught conceptually without a mathematical underpinning. Upon completion of the science degree, she felt that if taught at all, QS should be in the context of the science discipline-specific courses, but only on a '*need to know*' basis.

Barnett and Coates [11], and Barton [12] have highlighted the critical role of 'being'. Barton discusses how a teacher's VPRO gets at their sense of being a mathematician, of 'holding mathematics' as a fluid part of who they are and how they see the world. In fact, Barton argues that a teacher needs to express their VPRO in the classroom and that students should be given the space to conceptualise and articulate their own VPRO. Barnett and Coates argue higher education curriculum has a role to play in developing students as people, that 'acting and knowing' must be conceived within the curriculum along with being; and hence the essence of 'acting, knowing and being' as a model for curriculum is about understanding their relationship. Midas gives us a picture of a curriculum where mathematics is considered in isolation of science, with first year units build mathematical knowledge in ways that engage and excite her as a person who enjoys mathematics. She sees units of study, we might suggest she sees units of knowledge. As she progresses into upper levels of science

units, the application of mathematics is reduced to basic, functional tools with no indication of learning, linking or applying exciting mathematics to experience the learning of science or the *becoming of a scientist*.

Conclusion and Implications

Barnett and Coates [12] postulate that not to consider all three aspects of curriculum - knowing, acting and being - results in fragmentation that leads to 'dangerous consequences', with the curriculum not achieving its intended outcomes for the learners. Whilst we can not draw broad implications from this singular case study, the analytical framework as applied to the story of Midas in regards to building QS provides support for the claim that disconnection of 'acting, knowing and being' in the curriculum can hinder the achievement of desired outcomes. Midas' story highlights how the 'QS-intended' curriculum did not achieve its stated aim for this student at this institution. Whilst the details of the 'QS-intended' curriculum were not discussed in-depth beyond the stated aim, the story raises questions about how the intended QS curriculum was planned and enacted. Indeed, an on-going government funded project into QS in Australian science higher education has revealed the extent to which higher education curriculum is left unplanned, with little evidence of the achievement of learning outcomes like QS (see www.qsinscience.com.au). In fact, just this year national standards for threshold learning outcomes of science graduates were articulated (<http://www.altc.edu.au/standards/disciplines/science>) and the government will be seeking evidence of their achievement in the coming years in Australia. It is worth noting these science learning outcomes are underpinned by QS.

The implications of this single piece of research raises the questions as to how the application of mathematics in science is planned, and enacted by university educators, and how it is experienced by students. A study into an interdisciplinary science-mathematics first year unit highlights the challenges faced by scientist and mathematicians as they work together to build an appreciation for QS in science [13]. Whilst only a single case study of a single student, the experience is supported by a quantitative study benchmarking the perceptions of learning outcomes from Biomedical Science graduates, which found that whilst students perceived the importance of QS in science, their confidence and notions of improvement of QS as results of their studies was the lowest rated outcome compared with skills like teamwork, writing and communication [14].

As more science higher education curricula aspire to build QS in response to the changing nature of modern science, research into the interdisciplinary design, development and implementation of science-mathematics curricula could provide beneficial. Current research in education is rooted in science or mathematics, with some research into interdisciplinary science-mathematics education in the school sector. The complexity of learning and teaching in mathematics is well recognised as are the prevalence of poor pedagogical practices in science undergraduate education. A new challenge is emerging with scientific bodies calling for greater levels of mathematics in science. Educational research has a contribution to make in this area, although new theories, methodologies and collaborations are needed within the context of science and mathematics in higher education. As this single case study attempted to demonstrate, theories and frameworks can be borrowed and adapted. The challenge will be linking research to practice to inform how curricula are designed where science and mathematics are interlinked meaningfully such that students experience many opportunities to act on the knowledge of mathematics in ways that influence their sense of being a modern day scientist. A starting off point is needed, a

place for scientist and mathematicians to consider the interdisciplinary curricula, and this is where research can initially contribute.

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What are our Senior Undergraduates of Mathematics Learning? A Mathematician's Hope

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This case study concerns the mathematical learning experience of a group of students attending a core third-year undergraduate pure mathematics module (Metric Spaces) in an Irish university. It emerged in response to a need to understand the barriers to accessing the level of abstraction required by the module and to a lack of relevant literature at the level of advanced undergraduates of mathematics. It was enabled by the teaching and learning opportunities afforded by a relatively small class, namely 30 students. This research explores how these students responded to an insistence on building (Metric Spaces) theory through guided self-discovery. This paper reports on the attitude towards, development of and facility with proof and proving in such advanced mathematics-major undergraduates.

Keywords: abstraction, proof and proving, theory building

Introduction

I learned things that I never learnt before. Metric Spaces is absolutely by far the subject where you learnt more from the point of view of making things myself, making proofs, starting up proofs, finding arguments, finding things....You went from that stage to being active rather than just sitting there taking in, taking in, taking in. You would have to do things yourself. Things a few months before you would think you couldn't do these things. Mathematicians do this kind of thing. You are a student; you don't do these things (Jessie, BSc).

This work arose as a result of two prior challenging years of teaching Metric Spaces as a core module to a heterogeneous class of, principally, third year mathematics-major undergraduates.

Students from the Financial Mathematics and Economics (FME) programme in particular typically form a sizeable majority (in the case-study year with 20 students from FME in a class of 30). The compelling evidence from the previous two years of teaching this module was of blanket disengagement from those within FME and, by dint of sheer strength of numbers, of this cohort exercising a certain stranglehold on the overall learning experience.

The subject itself has its roots in the real number system and offers significant scope for learning through self-discovery on the basis of a small number of fundamental building blocks (such as open set, closed set, closure, limit point etc). One can explore and progress conceptual grasp of notions of convergence by pursuing connections, via conjecture and ultimately proof or counterexample, between such building blocks. It is commonly accepted amongst mathematicians that such activity should be entirely within the reach of mathematics students at this advanced stage of their programmes. The essential ingredient is openness to and facility with proof and proving. Such mathematical growth is at odds with the well-practised and by now accomplished method of learning by rote, with a learned disinclination towards depth,

with a way that has been reinforced by drill, practice and notably examination success as witnessed at post-primary level in Ireland [4].

In order to try to improve the mathematical experience for all concerned, I sought to infiltrate the class as a whole through the implementation of a range of learning strategies which are described later. While such was the initial impetus for this pilot project, subsequent and continuing reflection suggests insight that goes beyond intra programme division and perceived disinterest (on the part of some students, at least) and strikes at the heart of enabling *real learning* at this level.

Teaching Strategies

It is pertinent that the subject area, Metric Spaces, is the students' first comprehensive encounter of an abstract analytic subject, which necessarily has a corresponding impact on their engagement and consequent learning experience. The module is of standard format, running for two (lecture) hours per week with a weekly supplemental tutorial and with continuous assessment. Students also carry five other modules in the same semester, at least two of which are within mathematics. As with most pure mathematics modules, this module is 279ormalizing279d by theory and proof, and so understanding and skills of reasoning are key. There is little opportunity for an automated procedural approach to flourish; rather the effort and the emphasis are on advanced mathematical thinking. My particular focus in teaching the module was to develop the students' competence and therefore confidence in proving results for themselves, indeed on 279ormalizing such an activity. I was less concerned with the 'big theorems' of the area but rather that students could (re)create lemmata and theorems on their own.

In an effort to mitigate a tangible groundswell of disinclination combined with the complication of a 'hard' subject area, I introduced some tactics to counteract at least some of the forces at play. The intention was threefold: to excise an unhelpful disposition in some so as to allow us all to get on with the business of mathematical thinking; to incorporate collaboration as a useful and indeed normal mathematical practice and skill, and to stimulate engagement through peer-led learning:

Goal	Strategy
Engage and energise students, FME students especially who were unconvinced of the module's relevance to them.	Play down the subject matter and instead 'talk up' the advanced mathematical thinking aspect in the guise of proving. Emphasise and value the transferability of skill.
Develop aptitude for building theory (proving) from small beginnings; encourage confidence in discovering results for oneself.	Confine scope of module to a limited number of central definitions and concepts from which to (self-)assemble the remaining module content; trade quantity of knowledge for quality.
Promote the existence of a valuable forum for both individual and collaborative group learning; promote the necessity of weekly and ongoing study. Dismantle divisions and promote commitment to work ethic.	Run three compulsory weekly tutorials with a fixed 10 students, decided by lecturer, per tutorial.

Give focus and purpose to tutorial sessions and stimulate further team meetings outside of formal contact hours. Promote general and whole-class discussion of problems. Cultivate a culture of mathematical effort and industry.	Introduce team-led approach to assignments: within a given tutorial group, students were assigned to teams that took ultimate responsibility for certain problems from a given problem set. Teams were selected judiciously with an eye on programme composition.
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In essence, the scheme was a reaction to my growing suspicion that, when the perceived blanket of indifference is lifted, there are significant issues for a significant number of students in terms of mathematical thinking. It seemed to me that many students actually didn't know how to study advanced mathematics for understanding nor even realise the significant time required in the pursuit; that they didn't know therefore the exquisite buzz of enlightenment, of triumph, after the hard graft of concentration and effort. Barton [2] highlights how *mathematicians speak like addicts about their subject, deep mathematical knowledge is a source of intense and intimate pleasure*. He suggests that the undergraduate mathematics experience might benefit from a design intended to get students similarly addicted. He writes that in order to get people addicted, classical behaviourist theory asserts that *intermittent, irregular reinforcement* is required.

This may be what research in mathematics provides for its practitioners – expanses of frustration, lots of work, and mazes of dead-ends followed by the rare thrill of a breakthrough. But it is not how most mathematics courses are designed. Students either get little or no reinforcement at all, or get it frequently and regularly for not much effort.
(Barton, 2010)

It is well documented [1] that progress in mathematical thinking is just not linear; it happens in fits and starts, with ideas and concepts beginning as difficult and awkward yet which, through regular revisiting and interrogation, gradually yield to the learner's grasp. This type of growth necessitates time and effort on an 'early and often' basis. The group initiative was an effort to inculcate such a practice. Formalising group activity in this way required greater individual responsibility on the part of many and ultimately greater engagement. Of course, there were students for whom no such action or methods were required. However, the overwhelming evidence from the two preceding years alerted me to the needs of many others. What I did not know was the extent to which the perceived level of engagement was a response borne of experience, or whether it masked a deeper unduly problematic state of mathematical development.

The final examination, accounting for 70%, remained predictable in nature, with questions covering the module content and associated assignments as usual. Theoretically, rampant rote learning could result in examination success but in practice it did not as the method buckled in the face of material ridden, albeit organically, with a demand for proofs. If stirred into study only in the run up to the examination, again a common practice, the fall was all the greater. In terms of study method, it seemed that the freedom of knowing and knowledge through understanding was considered no match for the burden of the old reliable rote learning with its tried and tested outcomes.

Research Design and Methodology

This paper reports on the following research question:

What does a typical third year student of mathematics understand as their role in the learning of advanced abstract mathematics? What constitutes learning in this regard?

I chose to explore this research initiative by means of a qualitative approach using audio-recorded semi-structured interviews of duration approximately one hour each. I adopted a case study design, with a grounded theoretic aspect.

Interview design

Concerning this research, there is no doubt that *an interpretive orientation flows through qualitative research* ([3]) and that considerations of philosophy and paradigm serve to mitigate unintentional prejudice. My position as both teacher *in* the case study and researcher *of* the case study necessarily gives cause for concern. I sought to minimize this conflict by distancing myself from the data collection aspect of the process as far as was possible. Further details concerning this important issue are described below. As to the design of the interview schedule, I sought to encapsulate the central research theme through a range of broad-based questions, as follows:

1. Can you remember what your perception was of *learning* mathematics back in the first year of your programme?
2. Has your perception changed in your time here? How and why?
3. Do you have any sense of *becoming* a mathematician through your degree programme?
4. What do you think is required of you in order to learn abstract mathematics?
5. Leaving aside exam and assessment results, did you have any other tangible sense of improving or progressing mathematically in the course?
6. As learning activities, how did you find the assignments for Metric Spaces?
7. Take a look at the following problem from Metric Spaces please; how would approach a solution to it?

By embedding in the interview schedule and admittedly poised at the end, a mathematical task relevant to the module, the intention was to turn the focus eventually round to doing mathematics as a natural progression from talking about doing mathematics. The task was varied so as to reduce the risk of one post-interview student discussing their task with a pre-interview student.

Data collection

Conscious of my position as lecturer for the module and of the ensuing power relationship that naturally exists, I secured as a key resource for the study the recruitment of an interviewer with a strong mathematics background and with exemplary communication and people skills. The interviewer was briefed on the way the module was taught, on the interview schedule and on the general purpose of the research.

Data collection commenced early in the second semester and took the form of semi-structured individual interviews as described above. Thus from mid-February, the interviewer invited all members of the former Metric Spaces class to participate. The timing of this ensured that I was removed from an obvious position of influence in that all assessment of this module had been completed by then. Our efforts netted a

sample of eight students from the possible 30, a sample which fortunately represented a broad spectrum of ability, experience and especially degree programme. Note in particular that the embedded mathematical task relating to metric spaces would be presented to the participants some two to three months after they had completed the module. Thus the enduring nature of such learning as they had accomplished during the module would be tested. In keeping with the central thrust of the research, the tasks were designed so as to be accessible without reliance on memory. They also would have been discussed in class. Thus all relevant definitions and other information were provided for the students; the tasks were designed so that the students would have to create a proof based on all relevant information being available to them. This would be the acid test in terms of mathematical progress in the matter of proof, a theme central to the module. The interviewing process was completed in April of that year. The data was subsequently transcribed by an independent third party.

Data analysis

Through repeated immersion in the data transcripts and audio-recordings, I undertook the gradual process of data fragmentation via open and then selective coding. It was possible to do this without the aid of data analysis software due to the relatively small sample. Note that the eight participants have been given pseudonyms in this account. Also the relevant academic year has been concealed throughout to further protect their identities.

Statement and Discussion of Findings

Participants

The students' accounts were obviously flavoured by their backgrounds and their expectations. In this regard, it is germane to note that two participants Jessie and Daragh had received European non-Irish post-primary education. Otherwise all eight participants had come through the same menu of mathematics modules from first year university in pursuit of a mathematics- major degree. Jessie and Daragh were in the penultimate year of a BSc programme, Chris and Pat were in the final year of a three-year BA programme while Alex, Matt, Sam and Jean were in the penultimate year of an FME programme.

All kinds of knowing

There is a problem with learning, and it starts with the word itself. Its meaning is ambiguous. The same remark applies to 'knowing'. Thus Jean, in speaking about studying school mathematics, says that she *would always know it from understanding it and not from learning it*. One infers that here *learning* is in fact 'learning off' whilst knowing (from understanding) is the highly desirable state we seek to inspire in all learners of mathematics. All six Irish students acknowledged the particular and indeed accepted role of memorization where it came to proofs in leaving certificate mathematics (the final post-primary Irish state examination), a practice which persisted in first year university mathematics. Sam asserts that *memorising is just for the exam whereas, with learning, you are developing yourself in your mind* while Chris offers that *definitely with maths, there are parts you have to memorize. Just to get past the exam*. There is a general view from these six participants that memorization can and does still function well at university: *to be honest even like in*

3rd year or in final year the whole learning off thing still goes on (Pat).

Its purpose however is solely and startlingly for examination success and thus it raises serious questions about the fitness of purpose of examinations at this level, at least as they have been experienced by these students. When asked about the role of memorization, Jean says

... if you are really stuck and you don't know what you are doing for the exam, it is handy that the theorems are there to learn off. So you can bail yourself out with knowing them.

Consider then the position the students find themselves in when faced with a module entirely made up of theorems (with the accompanying definitions, examples and proofs). The hallmark in learning such mathematics is to learn how to construct knowledge (theorems) from a bank of definitions and concepts, and to create internal connections. Even with limited interest in the subject matter itself, its power to enhance transferable skills of reasoning is significant. To develop such skills, it is necessary to be able to both understand given proofs and to exploit that understanding to derive proofs for oneself. Indeed this is what lecturers would believe and hope they are nurturing in teaching pure mathematics modules from first year onwards. But this study suggests that something is amiss.

In a lot of courses, people tend to leave it until the last month to start learning and you might still get very good grades in the exam but after a couple of weeks you have forgotten everything completely (Matt).

Jessie coins the term 'interior learning' to describe the learning that comes from working and thinking continuously over time, the type of learning that she recognizes as meaningful.

Expectations of and for learning

While each student acknowledged the desirability of understanding any mathematics they encountered, there was some variation as to what that might entail or indeed to what lengths each might be prepared to go. 'Studying' can cover a spectrum from the helpful but potentially superficial reading of lecture notes and other relevant material to 'interrogating' what is read by picking apart proofs and definitions so that they become internalized. Thinking is of course essential, all of which is time consuming and certainly responsive to engagement from the outset. Late starting courted disaster. Each student confirmed how different and difficult they had found the module. *It was probably the hardest subject in maths that we had done until then (Sam).*

I think that there is an issue of expectation here, on the part of both the lecturer and the student. As a lecturer who has gone further along the path in thinking mathematically, I am all too familiar with the effort required in conceptual understanding. I am also of the opinion that the foundation and development offered over the course of the preceding two years of pure mathematics should at least alert if not accustom students to the demands of mathematical thinking and to successful learning strategies. It is important to note that Metric Spaces is a typical module in a mathematics degree. It is not at all non-standard nor is it unreasonable to ask that students study it; hence my fascination and frustration at the paralysis that either I or it seemed to be causing. Matt's frank observation, having heard about the previous year's FME failure rate in the subject, is eye-opening:

From the beginning...I made sure I understood everything in every lecture and after

every lecture I went over it which I hadn't done with any other course. I definitely feel an understanding for Metric Spaces because of that. Other courses I can't even remember them now from last semester....but I feel Metric Spaces is still in my mind because I gave that probably nearly as much work as the rest of the courses combined last semester.....With all the other courses, you just go and read the notes and you can understand it....With Metric Spaces, I found if you missed something, then everything after didn't make sense.

Mathematical foundation

That is really an undertaking, you know, trying to do a proof. I didn't do that in second year, ever. We were just learning proofs (Jessie).

Given the distinct possibility of habits of superficial learning, and that a module like Metric Spaces builds upon the security of first year concepts, perhaps the signs were always there. The six Irish students referred to the bewilderment wrought of abstract first year notions and the coping mechanism of memorization. As mentioned earlier, Metric Spaces is an abstract analytic branch of Pure Mathematics. Before expounding upon its theory and power, the lecturer would typically revisit the parent example of the real line with which all students have some familiarity.

There is at least one obvious problem. If concepts, central at least to Metric Spaces and unpleasant if not confusing to first years, have not bedded in somewhat better during second year through the passage of time, through reinforcement and through developing mathematical maturity, then the foundation can only be weak. Jean makes exactly this point:

.....a certain level is expected of you when you go into the course when you don't really have it. It started off in second semester in first year to understand more abstract things and when you didn't know it then and you just got through first year....you are in trouble.

Continuity of functions is a key notion in mathematics. I used the move from the much maligned epsilon-delta version to more general (and abstract) open set and convergence characterisations as a focus for developing skills in proving and connecting, as well as illuminating the concept itself. Over time, students gradually yielded to the power of abstraction:

I understood what continuity meant at the end of Metric Spaces. We have been talking about continuity since secondary school but other than learning off a definition, I never really understood what it meant. Even at the start of Metric Spaces, it was still just a definition that I learnt off but once the second and third version of continuity came in, it just clicked what it meant (Matt).

To a significant extent, the time in the module was spent addressing foundational matters of definition and concept alongside facility with and for proving. Pat perceives

....you got a lot of basics really, really down whereas in other subjects you wouldn't have as much of a basis behind you...I just found that I knew where I was going. I knew what was going on.....What I really found is that it was very beneficial from the foundation point of view. In a way I would nearly say it would be better to take it in 2nd year but I don't know if someone in 2nd year would have the mathematical maturity to cope with it but I think there are arguments for it.

Thinking mathematically

The notion of *thinking* on an ongoing basis, as encouraged by assignments,

emerged as a central theme. The assignments were fairly consistent in terms of level of difficulty, with the emphasis always on proof-making. The intention was for basic 'atomic' results to be derived from the students' own thinking and proving, drawing only on a limited amount of content knowledge and logical skills. I underlined the adequacy of understanding only a few key definitions and concepts together with skills of reasoning in order to push through some connection via proof. I hoped to instill some level of self-confidence by favouring quality of thinking and understanding over quantity in terms of content. In essence, students were given a controlled set of 'building blocks' [Jessie] from which to create theory. The degree of containment and indeed security was noticed:

...you had very few central ideas that everything came from. There wasn't anywhere too far wrong where we could go which I think helped a lot because we couldn't get it really, really wrong..... You were building, building, building and you could see where you were going. I thought that was important.....: Very little was given to us straight up, you had to think about it first which I think is a good approach...(Pat).

Jessie offered a strong opinion on the assignments, an opinion reflecting Barton's reference to the nature of assessment:

..the problem sheets in Metric Spaces were very hard at first sight...you start working on the first question. You think and think and think and then you get to something.... I found them really good in the sense that they were the centre of the balance exactly. They weren't too easy, they weren't impossible whereas in other subjects maybe you have too easy things or really impossible stuff. Metric Spaces ones were really well calibrated...

Achievement and affirmation

While the data certainly highlighted the individual efforts given in the endeavour (to learn and achieve), there was also recognition of self-improvement. This points in the direction of self-monitoring, another concept mentioned by Barton [3]. This is an attribute associated with self-directed learning that must be nurtured; again, we are brought to the realm of expectation. I believe that students should expect to self-regulate in terms of their own learning, yet the data indicated that there is a strong desire for external pressure in this regard. It is likely that prior strong conditioning in being tested by others (post-primary) has developed a dependence on and expectation of such external forces.

Those elusive 'aha' moments that critically and thankfully buoy the spirits of many flagging professional mathematicians in their research were experienced by some:

It happened once or twice and I was so excited with myself when it happened.....it was like a really good feeling to prove it yourself because you felt like you really learnt something and you really knew stuff (Pat).

However, there is a seemingly unassailable gap between how a lecturer can help in terms of a student's understanding, with the resultant and empowering sense of accomplishment and confidence, and what the student needs to do for him or herself. Jean's sentiment, expressed below, conveys at least the sense of enjoyment she received from her efforts to engage with this module. Unfortunately, the frustration is scarcely concealed: *You love doing [Metric Spaces], you just wish you understood it a bit more.*

Others speak of confidence being instilled by the lecturer. Varying degrees of confidence were illustrated with the embedded mathematical tasks held during the

interviews. These were taken directly from the Metric Spaces coursework and so had been seen before. For convenience they are reproduced here:

Task 1: How could you show that the intersection of two open sets is open?

Task 2: How could you show that the union of two closed sets is closed?

Note again that all relevant information was provided as context, namely the definitions for open and closed subsets of a metric space, and recall that the students last thought about Metric Spaces some 3-4 months prior. Each participant was asked to think about one task only. Naturally, the interviewer attempted to set students at ease regarding the prospect of solving a problem under scrutiny and assured each student that our only interest was in the process of proving rather than in their success at proving. Nonetheless, the anxiety wrought of being asked to prove a result in such circumstances should not be discounted. Of the eight participants, five (Pat, Daragh, Matt, Sam and Chris) demonstrated confidence in their ability to prove by thinking through their particular task slowly and logically, as befits a student of advanced mathematics. Their approaches were characterised by reflection, some pauses and then acceleration when the proposed proving strategy seemed to gel. They engaged directly with the information given and showed confidence that the proof was a matter of applying logic directly to that information. See Figures 1 – 5 in the Appendix.

Of the other three approaches, two (Alex, Jean, both FME) were characterised by efforts to remember the task from before and by struggle to understand the task before them (Figures 6 and 7). The confidence shown was very low, accompanied by little ability to use the definitions given to gain access to a proof. The final student's approach (Jessie, BSc) was more confident, with engagement of the given resources but at a slower pace (Figure 8). Whereas the five students had almost completed the proof, this last student took the problem to preliminary stages of proof– indicative of either individual stages of mathematical growth or discomfort in the given situation or both.

Final Word

This case study has highlighted the significance of expectation and responsibility in terms of learning mathematics in the pursuit of a primary degree therein. Such expectation and responsibility concerns student and lecturer alike. That ‘thinking for oneself’ was cited so often in the data as a feature of the learning experience in the module is disconcerting. Pursuing a degree with mathematics as a major component surely should have mathematical thinking at its very core. The same holds true for proof and proving. These notions are central to pure mathematics and should be cultivated gradually from first year in university. It is startling to hear, from these students at least, that active self-composed proof and proving was beyond their realm of experience and indeed expectation. One has to wonder, and worry, about the scale of this phenomenon.

Acknowledgement

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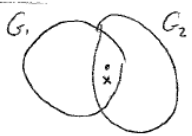
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Appendix: Figures

Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset G of a metric space (X, d) is *open* in X if for each $x \in G$, there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq G$ i.e. the open ball $B_\epsilon(x)$ is contained in G .)

How could you show that the intersection of two open sets G_1 and G_2 is open in X ?

Step 1: 

Step 2: $B_\epsilon(x) \subseteq G_1$ $B_\delta(x) \subseteq G_2$

Step 3: $B_{\min\{\epsilon, \delta\}}(x) \subseteq G_1 \cap G_2$

Hence, $G_1 \cap G_2$ open $\forall x \in G_1 \cap G_2$

Figure 1. Matt.


let G_1 be an open set in X .
 let G_2 be an open set in X

Since G_1 is open, for each $x \in G_1$, $\exists \epsilon > 0$
 st. $B_\epsilon(x) \subseteq G_1$.

Since G_2 is open, for each $x \in G_2$, $\exists \delta > 0$ st.
 $B_\delta(x) \subseteq G_2$

let $y \in G_1 \cap G_2$

Want to show that $G_1 \cap G_2$ is open. In other
 words, $B_\epsilon(y) \subseteq G_1 \cap G_2$



$y \in G_1 \quad \exists B_\epsilon(y) \subseteq G_1$
 $y \in G_2 \quad \exists B_\delta(y) \subseteq G_2$

$B_\epsilon(y) \subseteq G_1 \cap G_2$

Figure 2. Chris.

Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset G of a metric space (X, d) is *open* in X if for each $x \in G$, there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq G$ i.e. the open ball $B_\epsilon(x)$ is contained in G .)

How could you show that the intersection of two open sets G_1 and G_2 is open in X ?



Let $x \in G_1 \cap G_2$.

Let $B_\epsilon(x)$ be an open ball around x .

Figure 3. Sam.

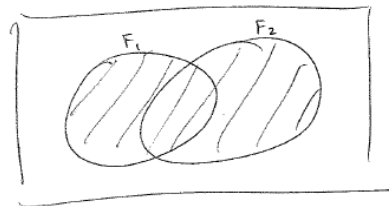
Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset F of a metric space (X, d) is *closed* in X if its complement $X \setminus F$ is open in X .)

How could you show that the union of two closed sets F_1 and F_2 is closed in X ?

RTFQ.

$X \setminus F_1$ & $X \setminus F_2$ both open



$$F_1 \cup F_2 = \cancel{X} \setminus (X \setminus F_1 \cap X \setminus F_2)$$

open ball in

Figure 4. Daragh.

Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset F of a metric space (X, d) is *closed* in X if its complement $X \setminus F$ is open in X .)

How could you show that the union of two closed sets F_1 and F_2 is closed in X ?

~~Pick any element that is in the union~~

Let $x \in F_1$ F_1 closed
 $x \in F_1 \cup F_2$ closed

Figure 5. Pat.

Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset G of a metric space (X, d) is *open* in X if for each $x \in G$, there is $\epsilon > 0$ such that $B_\epsilon(x) \subseteq G$ i.e. the open ball $B_\epsilon(x)$ is contained in G .)

How could you show that the intersection of two open sets G_1 and G_2 is open in X ?



Figure 6. Alex.

Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset F of a metric space (X, d) is *closed* in X if its complement $X \setminus F$ is open in X .)

How could you show that the union of two closed sets F_1 and F_2 is closed in X ?



Figure 7. Jean.

Activity

('Think aloud' as you write down ideas for a solution. Recall that a subset F of a metric space (X, d) is *closed* in X if its complement $X \setminus F$ is open in X .)

How could you show that the union of two closed sets F_1 and F_2 is closed in X ?


- ① closed set definition.
 - ② \cup two closed sets.
 - ③ Picture.
- 
- A Venn diagram with two overlapping circles, A and B. The intersection of A and B is shaded and contains a point labeled x.
- ④ Pick a point x
 - ⑤ You can see that it's closed.
 - ⑥

Figure 8. Jessie.

Tertiary Students' Diagrams for Reasoning About Proportions in Three Dimensions

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Students often use linear models incorrectly to reason about proportions in non-linear contexts such as area and volume. We investigate how the kinds of diagrams students draw can help them to notice cubic, rather than linear relationships when reasoning about proportions in three dimensions. We analyse the drawings of forty-six tertiary students who worked on a proportional reasoning problem set in the context of volume. The students were asked to divide up a catch of fish evenly by reasoning about the proportional change in volume of a fish whose width, length, and height were doubled. Our analysis shows that students' diagrams were developmentally linked to their understanding of proportional change in three dimensions. A developmental appreciation of student diagrams has implications for the kinds of advice teachers give students about using diagrams to reason mathematically.

Keywords: Diagrams, proportional reasoning, non-linear reasoning, tertiary mathematics education

Introduction

Experienced problem-solvers know that drawing diagrams can make their work easier [1, 2]. A good diagram identifies important information in a problem and arranges that information visually. Sometimes, the visual arrangement of the information will highlight the structure of the problem so well that it almost makes the solution appear obvious. Yet, students seldom use diagrams productively – they either draw superficial diagrams that fail to capture important information, or they simply avoid using diagrams altogether [3]. We propose that the kinds of diagrams students draw are directly related to the level of mathematical structure that the students perceive in the problem: Students who perceive surface level detail are more likely to draw diagrams that show surface-level information, whereas those who see deeper levels of mathematical structure tend to draw diagrams that are mathematically richer.

We demonstrate this relationship by analysing the mathematical structure evident in diagrams that 46 tertiary students drew while working on a proportional reasoning problem. We chose to set our study in the context of proportional reasoning as it is a critical foundation for many advanced concepts in mathematics, and yet proves to be a difficult topic for many students. Our results show that students draw diagrams that are appropriate to the level of mathematical understanding they have of proportional reasoning. This developmental approach to the drawing of mathematical diagrams has implications for teaching. Instead of simply telling students to “draw a diagram”, our results suggest that teachers should encourage their students to draw diagrams that are appropriate to their current level of mathematical understanding, but which push them to perceive a deeper level of structure in the problem.

An Overview of Proportional Reasoning Problems

The concepts of proportional reasoning have vast applications in mathematics, science and everyday life. Proportional reasoning has been described as:

... the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites.” [4]

Yet, despite being such a central topic in mathematics, research has shown that adults and students have persistent misconceptions about proportional reasoning [5].

One of the most common misconceptions is using *additive* reasoning in cases where *multiplicative* reasoning is called for. For example, in a classic problem [6], students are introduced to Mr Short, whose height is equal to 6 paperclips or 4 buttons, and Mr Tall, whose height is equal to 6 buttons. Students are asked to work out Mr Tall’s height in paperclips. Students often use *additive* reasoning in these types of problems, saying that since there is a two-button difference between Mr Tall’s and Mr Short’s heights, Mr Tall must be $6 + 2 = 8$ paperclips high. Instead, the correct solution is to use *multiplicative* reasoning, to see the relationships within and between paperclips and buttons as ratios, and thereby reason that Mr Tall must be $6 \times \frac{6}{4} = 9$ paperclips high.

Another common misconception is to use *linear* kinds of proportional reasoning for problems that are set in 2 or more dimensions. For example, when students are shown two rectangles, where one rectangle has length and width dimensions that are three times as big as the other (see fig. 1), students often incorrectly conclude that the area of the larger rectangle is 3 times as big as the smaller rectangle, when in fact it is $3^2 = 9$ times as big. Similarly, when students are given two cubes, one of which has length, width and height dimensions that are double the other (see fig. 1b), students often reason that the volume of the larger cube is twice as big as the smaller cube, when in fact it is $2^3 = 8$ times as big. Such linear reasoning is appropriate for situations that involve linear measures like length (as in the Mr Tall and Mr Short problem), time and money, but is not appropriate for problems set in two or more dimensions. Instead, students need to apply *non-linear* forms of proportional reasoning, such as reasoning using *quadratic* or *cubic* patterns.

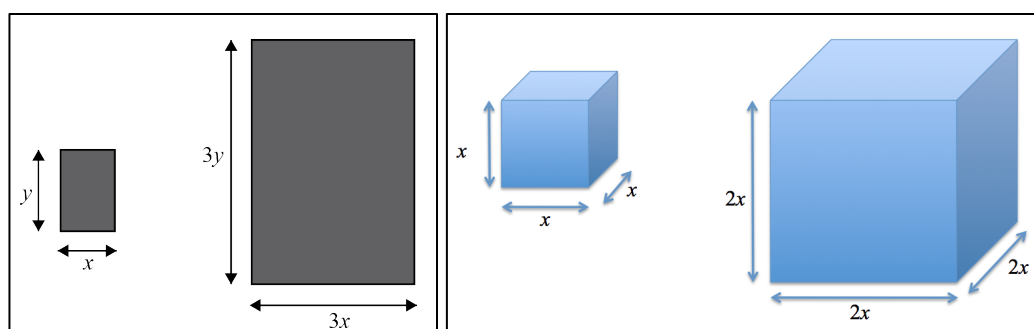


Figure 1. (a) A problem that calls for reasoning about proportions in two dimensions. (b) A problem that calls for reasoning about proportions in three dimensions.

In this paper, we focus on a problem that is set in the context of a fishing trip, which calls for proportional reasoning in three dimensions. The problem begins by pointing out that many marine organisms grow proportionally (see Figure 2a), in contrast to humans, whose head, body and limb proportions differ between babies and

adults.

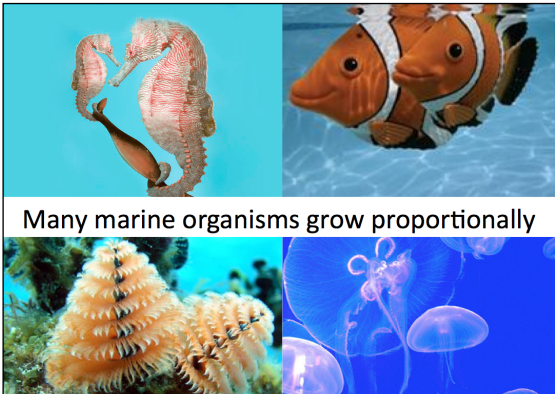


 <p>Many marine organisms grow proportionally</p>	<p>This is Squidley:</p>  <p>Squidley lives in the sea, and like many other marine animals, he grows proportionally.</p> <p>When he grows up he will be twice as big as he is now. What will Squidley look like?</p> 
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Figure 2. (a) Students are shown a slide with some marine organisms that grow proportionally. (b) Squidley warm up.

Next, students are introduced to a new marine organism called *Squidley*, which is made up of four multilink cubes, and are asked what Squidley will look like when he is twice as big as he is now if he grows proportionally. Finally, they are introduced to the problem statement:

The Loverich family and the Borich family went fishing together. They caught nine Snapper. Zoe Borich caught the biggest one, which was 54 cm long. Joe Borich took the job of dividing the fish up fairly between the two families so that they had the same amount of fish each. He gave himself the big snapper as his daughter Zoe caught it, and said that it was worth two of the smaller fish (27 cm each). Peter Loverich thought that the flesh from the big fish was probably more than four times that of the smaller ones but decided not to say anything to avoid a scene.

Peter needs your help!

Your job is to work out a mathematical argument for deciding how many little fish the big fish is worth, if the fish grow proportionally. **Write a letter** to Peter, describing your mathematical argument clearly, using diagrams if you wish. Peter wants to be able to use your argument for future fishing trips, so explain in your letter how he can make your argument work for fish of any size.

This problem calls for students to reason about the proportional change in the volume of fish when the fish's three dimensions of length, width and height are changed. Students initially tend to reason about the problem linearly, saying that since the length has doubled, the volume will have doubled also. They are encouraged to test out this linear reasoning as they work on the problem, and to adopt a way of thinking that appreciates that the volume changes in a cubic relationship with the scale factor.



Figure 3: Students were shown photos that showed a fish's width, length, and depth.

Literature Review: Drawing Diagrams as an Instructional Approach

A number of instructional approaches have been used to try to correct students' tendency to misapply linear proportional reasoning in situations that call for non-linear reasoning. These attempts include using metacognitive prompts to encourage students to reflect on the difference between linear and non-linear strategies [7, 8], setting non-linear proportional reasoning problems in real world contexts that students can relate to [9], and using diagrams as visual scaffolds. As our study looks at the role of diagrams in students' proportional reasoning, we review two studies that use the latter approach to try and teach students to apply non-linear proportional reasoning where appropriate.

In one study, a group of researchers gave students word problems that asked them to reason about the area of proportional shapes whose dimensions had been scaled by a common factor [7]. Half of the students were given drawings of the problem situation on squared paper, which were intended to goad them into noticing that the areas of the shapes changed as a quadratic function of the scale factors, rather than using incorrect linear reasoning. However, the researchers found no significant difference in the reasoning between students with the visual scaffold and those without. Moreover, they discovered that those students who were given visual scaffolds and performed better on the non-linear problems started to perform worse on linear problems, as they started to overgeneralise non-linear models to linear situations.

In a follow up study, the researchers [9] instructed students to generate their own diagrams to solve non-linear proportional problems, rather than providing them with ready-made diagrams as they had previously done. This instruction yielded surprisingly negative results – students who were asked to make drawings performed worse than those who weren't. De Bock et al. [9] proposed that the instruction to draw a diagram might have actually led students into the trap of linear proportional reasoning rather than protecting them against it: When students drew a reduced copy of a geometrical figure, they would have begun by measuring a linear element of the figure, such as the height or length, then would have divided that element by a linear scale factor. Thus, the act of drawing a diagram may have activated a linear thought process in the student's mind, rather than the quadratic or cubic kind of reasoning that the problem called for.

These two studies suggest that merely giving students diagrams, or telling them to draw diagrams is not enough to counter students' tendency to misapply linear proportional reasoning in situations involving area and volume. These findings are supported by more general research on the use of diagrams in mathematical problem solving, which suggests that students often do not know how to use diagrams

productively. Students often fail to look through diagrams they are presented with and see the structure of the problem that the diagram is intended to reveal [10], and often do not realise that diagrams are useful as, although their teachers often use diagrams in their own practice, students seldom appreciate diagrams' worth [11]. Students may draw ineffective diagrams or diagrams they think the teacher wants to see, rather than trying to focus on the relationship between the problem and the diagram [12]. While acknowledging the overwhelming evidence that students often use diagrams incorrectly, we propose that some student-generated diagrams may be useful, particularly when the diagrams reveal mathematical structure that is accessible to the students.

Conceptual Framework: Diagrams in Mathematical Problem Solving

A diagram is defined as “a visual representation that presents information in a spatial layout... [and considers] structural representations in which the surface details are not important” [10]. Diagrams “exploit spatial layout in a meaningful way, enabling complex processes and structures to be represented holistically” [13], and can “guide and constrain the range of cognitive actions” students use in a problem [10]. In this paper, we use Pantziara et al.'s [10] definition of a diagram, with its emphasis on the *structural* representation of information. We consider the types of information that students choose to include and omit and the way they arrange that information spatially in their diagrams as an indication of the mathematical structure they perceive. We restrict our focus to analyse the diagrams that students construct on their own as individuals, rather than the ones they were presented (by a peer or a teacher), and we focus on the diagrams that students drew to communicate their final understanding of the problem, rather than those they drew while developing that understanding.

Methods

The Fishing Trip problem was implemented in four classes at a large New Zealand tertiary institution. The participants were 57 students enrolled in foundation studies – that is, they were preparing academically for tertiary degree studies. The students worked in groups of three on the fishing activity during a 90-minute class. A researcher and the class lecturer were present in the classroom for each implementation, and interacted with the students to facilitate group discussion, and encourage them to use physical blocks and drawings to test out their ideas. The students presented their group solutions during the 90-minute class, and all students were exposed to different ways of reasoning that showed that one fish of length 54cm was worth eight proportional smaller fish of length 27cm. After the session, the students then had to complete, in their own time, written individual solutions to the Fishing Trip problem. Forty-six students handed in their individual solutions a week after the 90-minute class.

We collected the 19 groups' written work, the 46 individual written solutions, and researcher field notes of the student presentations. We analysed the individual solutions to assess the effectiveness of the mathematical argument, taking into consideration the written language, diagrams, tables, numerical examples, and algebraic expressions they used. We used the criteria in Figure 4 to assess the quality of their mathematical argument, focusing on the extent to which the letter provided an argument that would be sufficiently convincing to “Joe” (one of the characters in the problem). Two researchers independently classified the 46 individual solutions. Then,

they met with a third researcher, who arbitrated on cases where the classifications were different, and came to a resolution.

<p>Level 5: Convincing, correct, and generalized mathematical argument <i>"This is a convincing (and correct) mathematical argument that I could use with Joe. I could also use it to argue about the fair division of fish in any other fishing trip, not just this one, and with fish of other size."</i></p> <p>Level 4: Convincing and correct mathematical argument, but not generalised <i>"I can use this argument to convince Joe that he needs to give me 8 fish for the 1 big fish. But it won't help me divide up the rest of the catch."</i></p> <p>Level 3: Weak mathematical argument, correct final answer <i>"I can kind of see why you think I need to be given 8 fish, but I don't think Joe will be fully convinced."</i></p> <p>Level 2: No mathematical argument, correct final answer <i>"You've told me that I need to be given 8 fish, but there's no justification why. I need an argument, not just a solution."</i></p> <p>Level 1: Incorrect mathematical argument, incorrect final answer. <i>"Your argument doesn't make sense. Joe would never agree, and I don't think it gives me the correct amount of fish either."</i></p>
--

Figure 4. The criteria used to assess the quality of the mathematical arguments.

We identified three main types of diagrams that students drew: fish diagrams, which resemble the physical outline of a fish; block diagrams, which portray blocks or cubes that students used to model the fish; and fish-block diagrams that depict a diagram of a fish that has been divided up into equal parts or “blocks” to show eight equal parts (see fig. 5). We also analysed the accuracy of the diagrams and the extent to which a diagram was linked to the student’s overall argument. Excel was used to analyse the distribution of diagram types and features across the five levels of mathematical arguments.

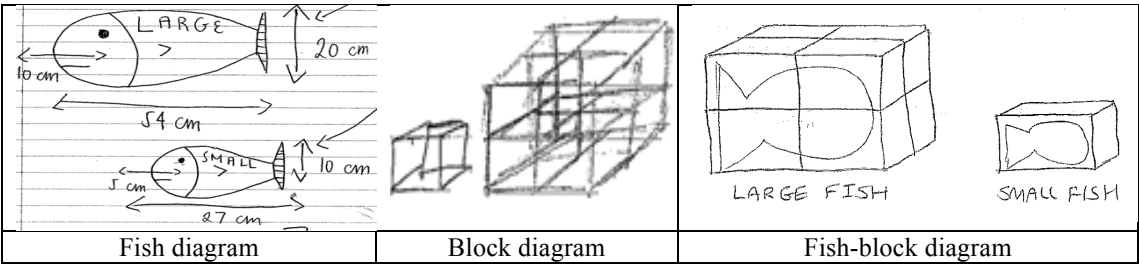


Figure 5: Three main types of student diagrams.

Results

The graph in fig. 6 shows the distribution of diagram types across the five levels of mathematical argument. It should be noted that the percentages within each level do not necessarily add up to 100 as some students drew more than one diagram – for example, a student may have drawn both a fish and fish-block diagram. Three significant patterns emerged:

1. Fish diagrams were often drawn to support arguments at levels 4 and 5
2. Block diagrams were drawn most frequently in arguments at levels 2 and 3
3. Fish-block diagrams were drawn at all levels of argument (except level 1)

Many students also used fish diagrams in level 1 arguments, but we will not report on these as they were not linked to correct reasoning about proportions in three dimensions.

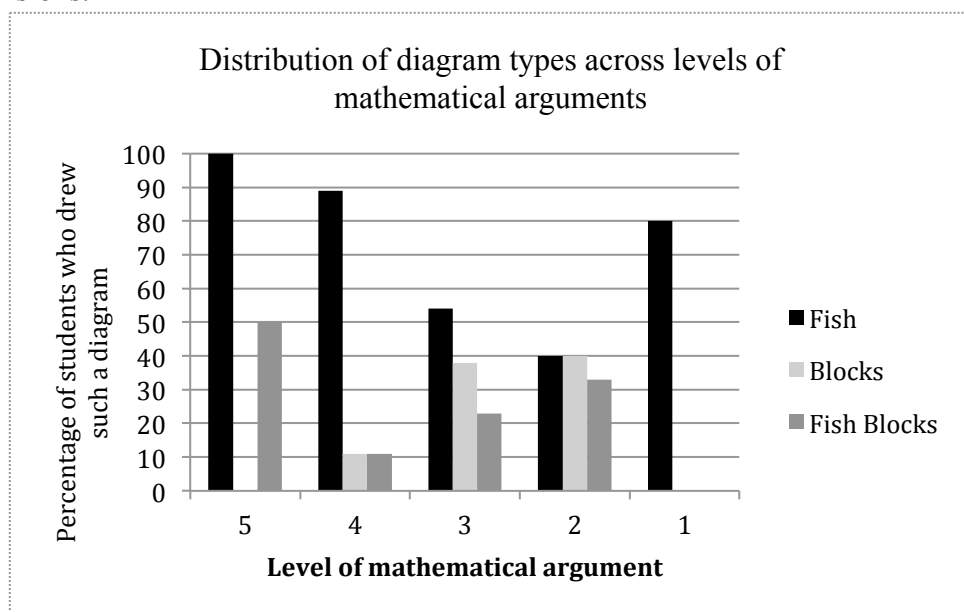


Figure 6: A graph showing the distribution of types of diagrams across levels of mathematical argument.

Fish diagrams were used to support arguments at levels 4 and 5

Fish diagrams featured predominantly in top-level arguments (levels 4 and 5), and were typically used to show hypothetical measures of the length, width and height of a “small” fish and a “big” fish with dimensions twice as long (see fig. 7).

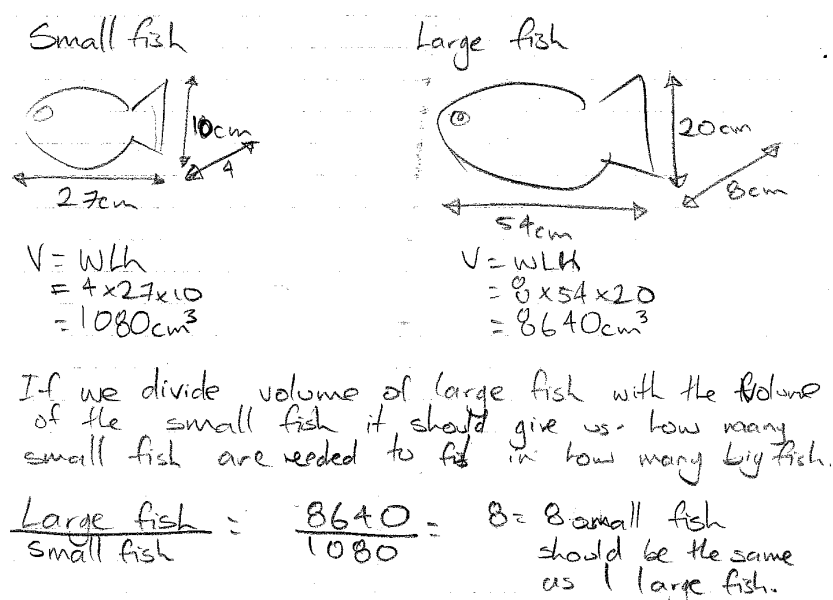


Figure 7: Fish diagrams in an excerpt from a level 4 argument.

In these instances, students often drew 2-dimensional representations of a small and big fish, and indicated the fishes’ three dimensions with labels and arrows. The students used these hypothetical measures to model the volume of a small and big fish, and divided the second by the first to show that the volume of the big fish was

eight times bigger.

Students who use fish diagrams to support their numerical examples in this way often draw diagrams with inaccurate proportions, thereby not accurately reflecting the difference in scale. For example, the drawings in fig. 7 do not reflect that the small fish is half the length, height and width of the big fish – indeed, the small fish appears to have length and width that are more than two thirds that of the big fish, and the third dimension (height or depth) is not drawn into the diagram.

Block diagrams often accompanied arguments at levels 2 and 3

Block diagrams were used predominantly in lower level arguments to model small and large fish. These were usually drawn quite accurately, sometimes with the help of a computer, and generally used a 3-D perspective drawing to show all three dimensions of length, width and height (see fig. 8).

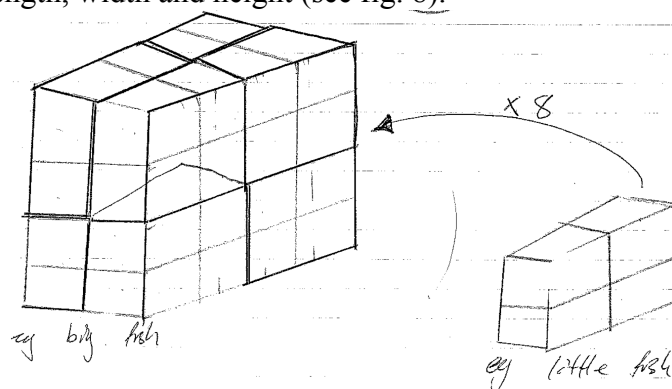


Figure 8. A block diagram in an excerpt from a level 3 argument.

Most students who used block diagrams did not use the blocks to explain clearly why doubling the length, height and width of the fish led to the volume being 8 times as big, and none of the students who used block diagrams were able to generalise their solution to other scale factors. Instead, most of the students who used block diagrams demonstrated that their diagrams showed one big fish was the same size as 8 small fish by counting the number of small “fish” (or collections of blocks) that fit into the large “fish”. In these instances, the act of demonstration through counting served as their argument.

Fish-block diagrams accompanied arguments at all levels (except level 1)

Fish block diagrams appeared in arguments ranging from level 2 to 5. Sometimes, they depicted fish as occupying the space of a block or blocks (as in fig. 5); other times, they depicted a fish carved up into separate regions the size of small fish (as in fig. 10). Sometimes, these regions were drawn accurately to show the true proportions of small and big fish; other times, the small fish were drawn inaccurately, and their positions within the large fish were more important than the size of the spaces they occupied (as in fig. 9).



Figure 9. Fish-block diagram in an excerpt from a level 2 argument.

Drawings that accompanied lower-level arguments typically used the fish-block diagrams in the same way students used block diagrams – to count the number of small fish within the large one. In contrast, some fish-block diagrams used the three-dimensional depiction to explain why doubling the length, height and width led to the volume being 8 times as big (as in fig. 10).

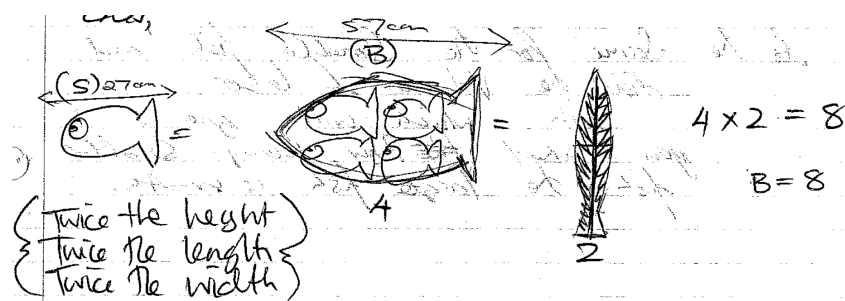


Figure 10. A fish-block diagram in an excerpt from a level 5 argument.

Discussion and Conclusion

Our analysis shows that students drew diagrams that were linked to the quality of their mathematical reasoning. Those at levels four and five who could explain why doubling the length in three dimensions yielded a volume $2 \times 2 \times 2 = 8$ times bigger often did not bother to draw their diagrams accurately. Instead, they mostly used diagrams to show hypothetical measures of the fishes' three dimensions, and the value of the measures on these diagrams were considered more important than the scale accuracy of the drawings. In contrast, those at levels 2 and 3 relied heavily on the accuracy of their block diagrams to show that eight small fish were equivalent to one big fish. In doing so, they avoided having to explain *why* eight small fish are equivalent to one big fish. Instead, this responsibility was offloaded onto the accuracy of the diagram: if the diagram was drawn accurately, they could simply count the number of small fish that fit into the large fish, without having to explain why it was so. Students used fish-block diagrams in both ways: sometimes to aid their explanation (by reasoning about the relationships between and within the three dimensions and volume), and sometimes to avoid explanation (by merely counting the number of small fish-blocks).

The students' diagrams in levels 2-5 suggest a developmental sequence that corresponds to deepening levels of mathematical understanding about proportions in three dimensions: Block diagrams \rightarrow fish diagrams, with fish-blocks scaffolding the transition from counting the number of fish to reasoning about the relationships between and within the three dimensions and volume. Note, we did not consider the diagrams drawn in level 1 arguments within this developmental sequence as they typically incorrectly involved proportional reasoning in one or two dimensions, rather than three dimensions. In general, teachers could calibrate the kinds of diagrams they encourage their students to draw to the level of mathematical structure the students are prepared to see. If students cannot see that one big fish is equivalent to eight small fish, then a teacher could advise them to draw diagrams that model the fish with blocks, and count them to verify that there are indeed eight. In order to have students begin to reason why doubling each dimension yields a volume eight times bigger with the irregular shapes of fish, teachers could facilitate the use of fish-block diagrams. Finally, we could encourage students to reason more abstractly about proportional relationships by moving away from discrete representations of blocks and fish-blocks

that enable students to base their argument on counts rather than proportional reasoning, and encouraging students to use diagrams of fish where counting is more difficult.

We have demonstrated that the diagrams students draw in one kind of proportional reasoning task reveal different levels of mathematical structure, which are linked to different levels of mathematical understanding. We propose that student drawings in other types of mathematical tasks may also exhibit developmental levels of mathematical structure. A potentially fruitful area for further research would be to investigate how students could develop their understanding of other difficult mathematical topics using diagrams that show a level of mathematical structure appropriate to their existing levels of understanding.

Acknowledgements

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Using Computer Algebraic Systems to Compute Antiderivatives: Showing Some Mathematical Facts That Should not be Neglected

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In this article we use different computer programs provided with Computer Algebraic Systems in order to compute antiderivatives of functions. The examples discussed here show that the use of technology can play an important role in observing and exploring mathematical concepts and theorems related to the Fundamental Theorem of Calculus, but mainly, they can be used as means to discuss important mathematical facts often neglected.

Keywords: Computer algebraic systems; antiderivatives; fundamental theorem of Calculus

Introduction

The Computer Algebra Systems (CAS) are tools that allow us to manipulate algebraic expressions, plot functions and operate with numbers. In the teaching of mathematics, it is well recognized that they offer the possibility of reducing the amount of boring and repetitive drill exercises and allowing for an increase in time spent, both in the classroom and elsewhere, on interesting and motivating aspects of the subject [2], [7], [8].

Thanks to the work of mathematicians and software designers, CAS are available nowadays on calculators and computers. Besides, currently there are free programs online and free sites where we can submit queries and computation requests of mathematics via a text field. In the case of Differential and Integral Calculus, for example, there exist computer programs especially powerful and helpful for addressing specific tasks.

In this article, we will discuss particular examples of computing antiderivatives with different programs provided with CAS. Attention must be paid to important mathematical facts often neglected in text books and in the classroom. In this paper, we will work with the computer programs: *Derive 6.0*, *Scientific Work Place 5.5*, *Mathematica 8.0*. And also with the free access site: Wolfram Alpha ([12]).

Using CAS to Compute Antiderivatives

In basic courses of Calculus, it is common to consider integration as the reverse of differentiation. Given a function f , the problem is to find another function F whose derivative is f . Such a function F is called an *antiderivative*. This general idea is due to connection between the processes of integration and differentiation established by the Fundamental Theorem of Calculus (FTC):

Theorem 1 (FTC):

Let $f:[a,b] \rightarrow \mathbb{R}$ be a continuous function. Let $F:[a,b] \rightarrow \mathbb{R}$ be a function defined

by

$$F(x) = \int_c^x f(t) dt \text{ with } c \in [a, b]$$

Then

F is differentiable for all $x \in [a, b]$ and furthermore $F'(x) = f(x)$ for every $x \in [a, b]$.

If G is a differentiable function such that $G'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) dx = G(b) - G(a).$$

The above version of the FTC is usually presented in a first year calculus university course (general forms of this theorem can be found in [1], [4] and [11]).

It is a common practice amongst students to compute an antiderivative F of a function f , and then to write the symbol

$$\int f(x) dx = F(x) + C$$

with $C \in \mathbb{R}$, to represent the set of all antiderivatives of f .

There are some standard techniques for computing antiderivatives which are applicable to several classes of functions. We can find in basic books of Calculus entire chapters devoted to those techniques, usually called *Methods of Integration*, some examples are: Integration by substitution, Integration by parts, Integration of rational functions and the list goes on. Consequently, this is one area where the CAS is a genuinely valuable tool. For several years, in general, students of mathematics had to use large tables of antiderivatives, although these are known to contain several errors. However, CAS have efficient algorithms that make them particularly effective at computing antiderivatives. These algorithms are often deeply unknown to most mathematicians [3].

Getting more than one result

Example 1:

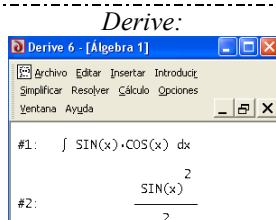
Find $\int \sin x \cos x dx$.

Indefinite integral:

Wolfram Alpha:

$$\int \sin(x) \cos(x) dx = -\frac{1}{2} \cos^2(x) + \text{constant}$$

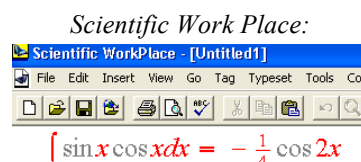
Derive:



#1: $\int \sin(x) \cdot \cos(x) dx$

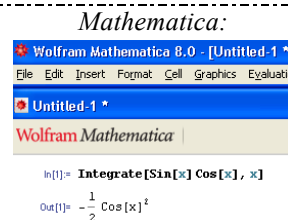
#2: $\frac{\sin(x)^2}{2}$

Scientific Work Place:



$$\int \sin x \cos x dx = -\frac{1}{4} \cos 2x$$

Mathematica:



In[1]:= Integrate[Sin[x] Cos[x], x]

Out[1]:= $-\frac{1}{2} \cos^2[x]$

Figure 1. CAS results.

As we can see in Figure 1, each program offers different answers. Considering this, we may write

$$\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2 = -\frac{1}{4} \cos 2x + C_3$$

where $C_1, C_2, C_3 \in \mathbb{R}$.

The first question arising here is: Are the different results given by the different programs correct? To answer this, a first strategy we can perform is to derive each result in order to determine whether we obtain the function under the integral symbol. We can easily verify that

$$\frac{d}{dx} \left(\frac{\sin^2 x}{2} + C_1 \right) = \frac{d}{dx} \left(-\frac{\cos^2 x}{2} + C_2 \right) = \frac{d}{dx} \left(-\frac{1}{4} \cos 2x + C_3 \right) = \sin x \cos x.$$

The next question is: how can educators explain to students the reason for the different results? In this case, getting different expressions of antiderivatives of the same function was expected due to the nature of the trigonometric functions (*i. e.* the existence of trigonometric identities) and it also depends on the strategy we have chosen. Evidently, each program has different strategies for dealing with Example 1. For instance, it is not difficult to see that *Scientific Work Place* uses the trigonometric identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

and then

$$\int \sin x \cos x dx = \frac{1}{2} \int \sin 2x = -\frac{1}{4} \cos 2x + C_3.$$

Fortunately, *Derive* and *Wolfram Alpha* have the *Showing Steps Option*, so we can see the strategy used. First, *Derive* uses the expression

$$\int F^n(x) F'(x) dx = \frac{F(x)^{n+1}}{n+1}$$

where $F(x) = \sin x$ and $n = 1$. On the other hand, *Wolfram Alpha* uses the expression

$$-\int u du = -\frac{u^2}{2} + \text{constant}$$

where $u = \cos x$.

Actually, the different results offered by CAS can be obtained without the use of it. However, this happens not only with trigonometric functions but in general as well. That is, if f is any continuous function, then it is possible to find more than one algebraic expression for an antiderivative of f . And this depends on the strategy followed; the only thing we need to ensure is that the different antiderivatives differ in one constant.

Theorem 2:

Let $I \subseteq \mathbb{R}$ be an interval. If F and G are two antiderivatives of f on I , then $G = F + C$ where $C \in \mathbb{R}$.

For example, considering the identities

$$\sin^2 x + \cos^2 x = 1 \quad \text{and} \quad \cos 2x = \cos^2 x - \sin^2 x$$

we get the following

$$-\frac{1}{4} \cos 2x = -\frac{\cos^2 x}{2} + \frac{1}{4} = \frac{\sin^2 x}{2} - \frac{1}{4} \quad \text{and} \quad -\frac{\cos^2 x}{2} = -\frac{1}{2} + \frac{\sin^2 x}{2}.$$

Which corroborates the fact that the antiderivatives provided by CAS differ from each other in a constant. Thus, it is correct to write

$$\int \sin x \cos x dx = \frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2 = -\frac{1}{4} \cos 2x + C_3$$

because each expression represents the same set of antiderivatives of f .

Now let's see another example where we need to be suspicious about the results given by CAS.

Example 2: Find $\int \frac{\cos 2x}{\sin x \cos x} dx$.

As we can see in the Table 1, once again we get more than one result. In this case there are three different algebraic expressions.

Table 1: Results of CAS for $\int \frac{\cos 2x}{\sin x \cos x} dx$

CAS	Result
<i>Derive</i>	#2: $\int \frac{\cos(2 \cdot x)}{\sin(x) \cdot \cos(x)} dx$ #3: $2 \cdot \ln(\sin(x)) - \ln(\tan(x))$
<i>Mathematica</i>	$\text{In}[1]= \text{Integrate}[\cos[2 \ x] / (\sin[x] \cos[x]), x]$ $\text{Out}[1]= \text{Log}[\cos[x]] + \text{Log}[\sin[x]]$
<i>Wolfram Alpha</i>	$\int \frac{\cos(2 x)}{\sin(x) \cos(x)} dx = \log(\sin(x)) + \log(\cos(x)) + \text{constant}$
<i>Scientific Work Place</i>	$\int \frac{\cos(2x)}{\sin x \cos x} dx = \frac{1}{2} \ln(2 - 2 \cos 4x)$

Without the use of CAS we can find another expression. That is, if we solve this example without the use of CAS can find other expression. For instance, using the identity

$$\frac{\cos 2x}{\sin x \cos x} = 2 \frac{\cos 2x}{\sin 2x}$$

and the formula

$$\int \frac{u'(x)}{u(x)} dx = \log |u(x)| + C$$

with the substitution $u(x) = \sin 2x$, we get

$$\int \frac{\cos 2x}{\sin x \cos x} dx = 2 \int \frac{\cos 2x}{\sin 2x} dx = \log |\sin 2x| + C.$$

Considering the above results, we may write

$$\int \frac{\cos 2x}{\sin x \cos x} dx = \begin{cases} 2 \log(\sin x) - \log(\tan x) + C_1 \\ \log(\cos x) + \log(\sin x) + C_2 \\ \frac{1}{2} \log(2 - 2 \cos 4x) + C_3 \\ \log|\sin 2x| + C_4 \end{cases}$$

for $C_1, C_2, C_3, C_4 \in \mathbb{R}$.

Again, a first strategy to confirm that the different results are correct is deriving the different expressions. We can also use the CAS to simplify this laborious task:

$$\frac{d}{dx}(2 \log(\sin x) - \log(\tan x) + C_1) = \frac{d}{dx}(\log(\cos x) + \log(\sin x) + C_2) = \frac{\cos 2x}{\sin x \cos x}$$

$$\frac{d}{dx}\left(\frac{1}{2} \log(2 - 2 \cos 4x) + C_3\right) = \frac{d}{dx}(\log|\sin 2x| + C_4) = \frac{\cos 2x}{\sin x \cos x}$$

Maybe we can think that the results are correct since, at first glance, the derivatives of the different functions are equal to the integrand. Nevertheless, in this example, we must consider the important fact that each function has a different domain.

On the one hand, the functions given by *Wolfram Alpha* and *Derive* $F(x) = \log(\cos x) + \log(\sin x)$ and $G(x) = 2 \log(\sin x) - \log(\tan x)$ have as domain the set

$$A = \{x \in \mathbb{R} : \sin x \cos x > 0\} = \bigcup_{n \in \mathbb{Z}} \left(2n\pi, \frac{4n+1}{2}\pi\right)$$

On the other hand, the functions given by *Scientific Work Place* and by hand computation

$$H(x) = \frac{1}{2} \log(2 - 2 \cos 4x) \text{ and } K(x) = \log|\sin 2x|$$

have as domain the set

$$D = \{x \in \mathbb{R} : 2 - 2 \cos 4x \neq 0\} = \{x \in \mathbb{R} : |\sin 2x| \neq 0\} = \bigcup_{n \in \mathbb{Z}} \left(\frac{n}{2}\pi, \frac{n+1}{2}\pi\right)$$

Let's recall the definition of antiderivative:

Definition 2

Let $I \subseteq \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be a function with I an interval. A antiderivative of f on I is any derivable function $F : I \rightarrow \mathbb{R}$ satisfying $F'(x) = f(x)$ for every $x \in I$.

Now, let $I \subseteq D$ be any interval and let $f : I \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{\cos 2x}{\sin x \cos x}$$

then, regarding the above definition, the functions F and G cannot be considered as antiderivatives of f due to the fact that they cannot be restricted to the interval

$I \subseteq D$. For example, although f is well defined on $I = \left(\frac{\pi}{2}, \pi\right)$, F and G are not.

In this case $I \not\subseteq A$. Hence, F and G are not antiderivatives of f .

Since H and K are defined on D and $H'(x) = K'(x) = f(x)$ for every $x \in I \subseteq D$, they are antiderivatives of f . Thus, it is correct to write

$$\int \frac{\cos 2x}{\sin x \cos x} dx = \frac{1}{2} \log(2 - 2 \cos 4x) + C_1 = \log|\sin 2x| + C_2.$$

The only one left is to verify that the functions H and K differ in one constant. Although it may offer a great opportunity to review some trigonometric relationships and properties of logarithms, it is not a simple task.

Getting a function not defined in the real numbers.

Example 3:

Find $\int \frac{\tan x}{\log(\cos x)} dx$.

As we can see in Figure 2, CAS offers us the same answer

$$\int \frac{\tan x}{\log(\cos x)} dx = -\log(\log(\cos x)) + C.$$

Let's analyze this result. Consider the function $u(x) = \log(\cos x)$ whose domain is the set

$$A = \{x \in \mathbb{R} : \cos x > 0\} = \bigcup_{n \in \mathbb{Z}} \left(\frac{4n-1}{2}\pi, \frac{4n+1}{2}\pi \right).$$

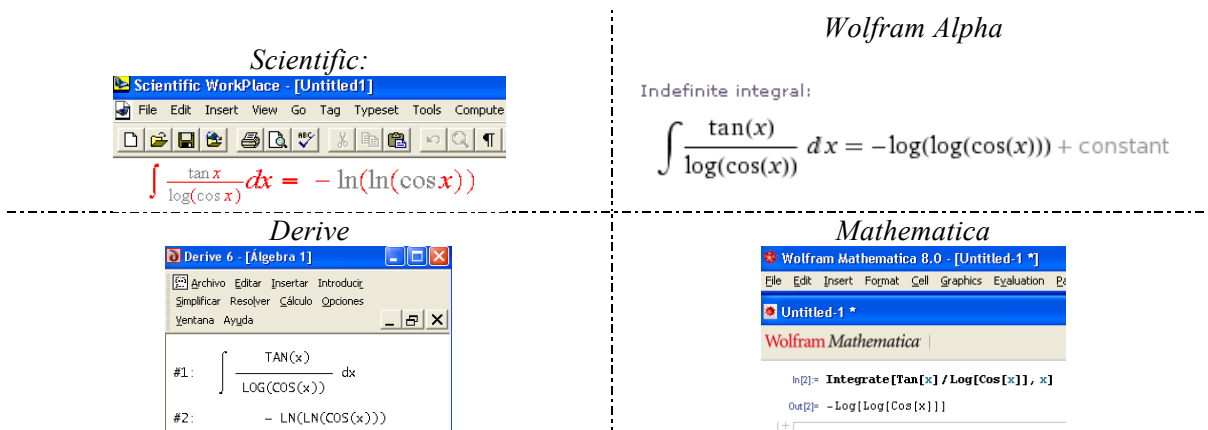


Figure 2. Results of CAS for $\int \frac{\tan x}{\log(\cos x)} dx$.

It is not difficult to see that $u(x) < 0$ for every $x \in A$. Hence, the domain of the function defined by

$$F(x) = -\log(\log(\cos x))$$

is the empty set.

Now, the domain of the function $f(x) = \tan x / \log(\cos x)$ is the set

$$D = \left\{ x \in \mathbb{R} : \cos x > 0 \text{ and } \log(\cos x) \neq 0 \right\} = \bigcup_{n \in \mathbb{Z}} \left(\frac{4n-1}{2}\pi, 2n\pi \right) \cup \left(2n\pi, \frac{4n+1}{2}\pi \right).$$

The function f is continuous on its domain. The FTC states that there exists an antiderivative defined on any interval $I \subseteq D$. Maybe the antiderivative cannot be expressed in terms of elementary functions (i.e. involving polynomials, and the standard functions sin, cos, exp, and so on). Notwithstanding, this is not the case, since we can find an elementary antiderivative. We just need to use the formula correctly:

$$\int \frac{u'(x)}{u(x)} dx = \log(u(x)) + C$$

which is an equivalent formula used by CAS (see Figure 3).

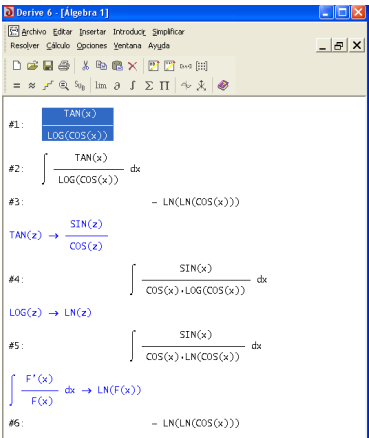
<i>Derive</i>	<i>Wolfram Alpha</i>
	<p>Indefinite integrals:</p> $\int \frac{\tan(x)}{\log(\cos(x))} dx = -\log(\log(\cos(x))) + \text{constant}$ <hr/> <p>Possible intermediate steps:</p> $\int \frac{\tan(x)}{\log(\cos(x))} dx$ <p>For the integrand $\frac{\tan(x)}{\log(\cos(x))}$, substitute $u = \log(\cos(x))$ and $du = -\tan(x)dx$:</p> $= -\int \frac{1}{u} du$ <p>The integral of $\frac{1}{u}$ is $\log(u)$:</p> $= -\log(u) + \text{constant}$ <p>Substitute back for $u = \log(\cos(x))$:</p> $= -\log(\log(\cos(x))) + \text{constant}$

Figure 3. Showing steps of CAS.

It is important to notice that the above formula is valid when $u(x) > 0$. To be more precise, we need to consider the absolute value. Considering this, we get

$$\int \frac{\tan x}{\log(\cos x)} dx = -\int \frac{u'(x)}{u(x)} dx = -\log|u(x)| + C = -\log|\log(\cos x)| + C.$$

As we have observed, in this case, the answer offered by CAS is not appropriate.

Getting antiderivatives with CAS for computing definite integrals

Most calculus students might think that if one could compute antiderivatives, it would always be easy to compute definite ones. After all, they might think, the second part of the FTC says that one just has to subtract the values of the antiderivative at the end points to get the definite integral. In general, the formula

$$\int_a^b f(u) du = G(b) - G(a),$$

is considered as a practical tool to compute the definite integral of a function f for which an antiderivative G is known.

Example 4: Find $\int_0^{2\pi} \frac{1}{5+3\cos x} dx$.

A suggested method for this kind of definite integrals is the substitution $u = \tan \frac{x}{2}$ (see [5, p. 344], [6, pp. 195-196] and [13, pp. 107-108]). Using this method, we get the function

$$F(x) = \frac{1}{2} \arctan\left(\frac{1}{2} \tan \frac{x}{2}\right).$$

However, we cannot use it for computing the definite integral on the interval $[0, 2\pi]$, since

$$\int_0^{2\pi} \frac{1}{5+3\cos x} dx = F(2\pi) - F(0) = 0$$

when actually the correct answer is $\frac{\pi}{2}$.

Now, let's try to compute an antiderivative with CAS. For this example we have used only *Derive*, *Mathematica* and *Wolfram Alpha*. As we can see in Figure 4, the CAS offers us two answers. Thus, we may write

$$\int \frac{1}{5+3\cos x} dx = -\frac{1}{2} \arctan\left(2 \cot \frac{x}{2}\right) + C_1 = \frac{x}{4} - \frac{1}{2} \arctan\left(\frac{\sin x}{\cos x + 3}\right) + C_2$$

with $C_1, C_2 \in \mathbb{R}$.

We can verify that these results at least differentiate correctly. That is

$$\frac{d}{dx} \left(-\frac{1}{2} \arctan\left(2 \cot \frac{x}{2}\right) + C_1 \right) = \frac{d}{dx} \left(\frac{x}{4} - \frac{1}{2} \arctan\left(\frac{\sin x}{\cos x + 3}\right) + C_2 \right) = \frac{1}{5+3\cos x}$$

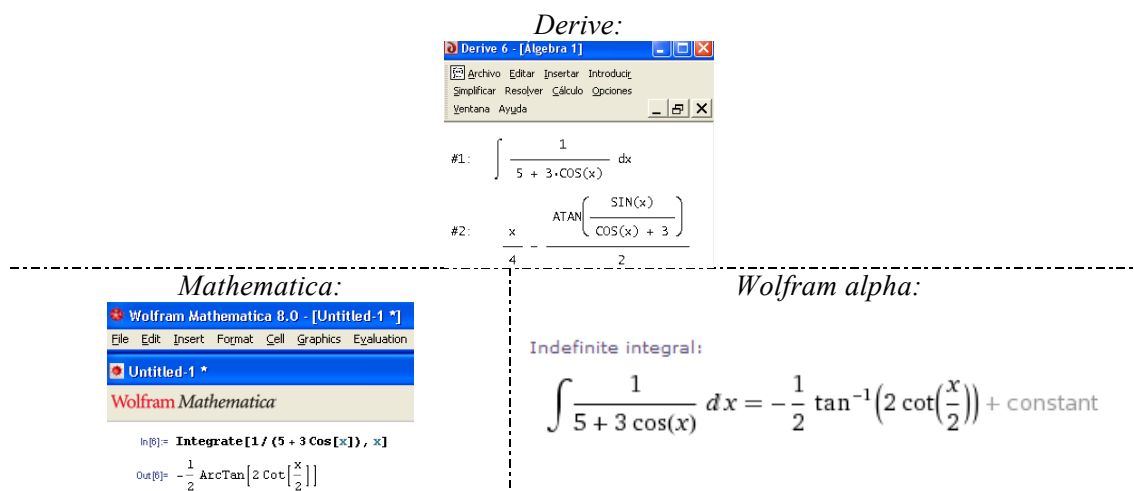


Figure 4. Results of CAS for $\int \frac{1}{5+3\cos x} dx$.

Once again, we need to consider some mathematical facts. First of all, the domain of the function f is \mathbb{R} . Now, on the one hand, consider the result given by *Mathematica* and *Wolfram Alpha*

$$H(x) = -\frac{1}{2} \arctan\left(2 \cot \frac{x}{2}\right),$$

whose domain is the set

$$A = \left\{ x \in \mathbb{R} : \sin \frac{x}{2} \neq 0 \right\} = \bigcup_{n \in \mathbb{Z}} (2n\pi, 2(n+1)\pi)$$

We cannot compute the definite integral since H is not defined in $x = 0, 2\pi$, that is, the expression

$$\int_0^{2\pi} \frac{1}{5 + 3 \cos x} dx = H(2\pi) - H(0)$$

makes no sense. Hence H is not an antiderivative of f . On the other hand, using the function

$$G(x) = \frac{x}{4} - \frac{1}{2} \arctan \left(\frac{\sin x}{\cos x + 3} \right),$$

given by *Derive*, we get the correct answer

$$\int_0^{2\pi} \frac{1}{5 + 3 \cos x} dx = G(2\pi) - G(0) = \frac{\pi}{2}.$$

Furthermore, the function G , is well defined on \mathbb{R} and we can restrict G to any interval $I \subseteq \mathbb{R}$. Thus, the function given by *Derive* is an antiderivative of f . Therefore, the correct answer is

$$\int \frac{1}{5 + 3 \cos x} dx = \frac{x}{4} - \frac{1}{2} \arctan \left(\frac{\sin x}{\cos x + 3} \right) + C.$$

We can also use CAS in order to compute the definite integral. In fact, within programs like *Mathematica* or *Derive* there are thousands of pages of code devoted to working out definite integrals instead of just subtracting the values of the antiderivative at the end points. However, in basic courses of Calculus, it is common to compute definite integrals via computing an antiderivative and then subtracting the values of the antiderivative at the end points.

We would like to finish this section making one last remark. Ponce-Campuzano and Rivera-Figueroa [10] pointed out that the classical method of substitution $u = \tan x/2$ fails for particular trigonometric functions. And it seems that *Mathematica* and *Wolfram Alpha* share the same problem. In regards to this, Elbaz-Vincent [3] and Pavlyk [9] have analyzed and provided explanations for the mathematical theories involved in the algorithms for computing antiderivatives of these kinds of functions, but they make some mistakes. For instance, Elbaz-Vincent [3] says:

But if f is continuous on I , so is G . However, an antiderivative is not necessarily an indefinite integral, and the CAS gives us several examples:

```
> int(6/(5-3*cos(x)),x);
3 arctan(2 tan(1/2 x))
```

Here the result is an antiderivative on \mathbb{R} , but fails to be continuous. As a consequence, we cannot use this antiderivative ‘out of the box’, in order to compute the definite integral of the function on $[0, 4\pi]$ for instance. The result would be nonsense (but apparently most students are not afraid of that) [p. 62].

In short, first what is being told, in the above quotation, is that ‘an antiderivative can be discontinuous’. This is clearly impossible because an antiderivative is always continuous (recall that an antiderivative of a function f on I is a derivable function

$H:I \rightarrow \mathbb{R}$ satisfying $H'(x) = f(x)$ for every $x \in I$ and a derivable function is always continuous). On the other hand, the particular example discussed by Elbaz-Vincent, that is, the function $F(x) = 3 \arctan(2 \tan(x/2))$, is considered as being a ‘discontinuous antiderivative’ of $f(x) = 6/(5 - 3 \cos x)$. However, neither F is an antiderivative of f nor F is a discontinuous function (recall that in the definition of continuity of a function, we consider only the points belonging to the domain: [1, pp. 160-161], [4, p.67], [11, pp. 141-144]).

Final Comments

The main point that we have tried to establish in this paper is that we cannot reasonably use a CAS as a black box, in particular in the classroom. As teachers, we should have sufficient knowledge of the behavior of CAS to understand the results for ourselves. And then we can encourage students to become more thoughtful about the use of CAS. The different examples discussed here show that the use of technology can play an important role in observing and exploring mathematical concepts and theorems related to the FTC, but mainly, they can be used as means to discuss important mathematical facts often neglected, in particular, when we compute antiderivatives of functions:

1. *We can get different algebraic expressions for antiderivatives.* In section 2.1, we have seen that, using CAS, we can get more than one algebraic expression for an antiderivative of a function. This certainly may happen not only with the use of the CAS but by hand computation too. The results of the CAS depend on the different algorithms based on mathematical theory, whereas hand computation depends on the different strategies we have chosen, for example: change of variables, integrating by parts, use of equivalent expressions, etc.
2. *The domain of antiderivatives should be taken in consideration.* Although, CAS is a valuable tool, it can give us inappropriate results. In the case of examples 2, 3 and 4, the most of the functions given by the CAS cannot be treated as antiderivatives, since such functions are not defined in the same domain as the integrand. Usually, we neglect paying attention to the domains of functions and this fact can lead to mistakes.
3. *The methods of integration usually fail.* We need to be careful when we apply any method to compute antiderivatives of functions because, in some cases, it can lead us to wrong answers; that is, we can get functions which are apparently antiderivatives. We have seen that the general formula

$$\int \frac{1}{u} du = \log|u| + C$$

is usually considered without the absolute value. On the other hand, the method of substitution $u = \tan x/2$ fails for particular trigonometric functions; however, in this case, *Derive* turns out to be a valuable tool since it gives us the correct answer when the mathematical theory fails.

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A Systematic Analysis of Errors in the Simplification of a Rational Expression

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Exploring the errors that mathematics students frequently make is a means by which teachers can gain a better understanding of students' difficulties. Reported here are the process by which the algebraic working of 95 undergraduate students who incorrectly simplified a rational expression was analysed and the results of the analysis. Initially, a deductive approach to analysing the errors was planned, categorising students' mistakes using the error types identified, named and described in the literature. In reviewing the literature, however, it became clear that this would be no simple task. The large body of literature, while rich in examples of "typical errors" that could be expected in students' working, had two limitations. Firstly, the error types lacked precise descriptions and were mainly described by example only. Secondly, insufficient details of the procedures used to categorise the errors prevented replication of the categorising process. Consequently, a mainly inductive approach, that categorised the errors by their location and inferred student operation was devised. This systematic approach resulted in generating descriptions of three error categories.

Introduction

Exploring the errors that mathematics students frequently make helps teachers gain a better understanding of students' difficulties. In articulating the benefits of approaching errors as a positive part of the learning process, Ashlock [1] writes (p.9): "As we teach computation procedures, we need to remember that our students are not necessarily learning what we think we are teaching; we need to keep our eyes and ears open to find out what our students are actually learning. We need to be alert for error patterns!" An important first step in analysing errors that students make is categorising those errors. Error categories help illustrate the patterns in the way students make errors. Furthermore, categorizing errors allows teachers and researchers to explore teaching strategies that may prevent or remedy errors that lie within the whole category.

In this paper we report on the findings of a study that investigated the errors undergraduate students made in simplifying a rational expression. The researchers anticipated using existing literature to categorise the errors observed. However, an extensive literature review found error categories that lacked the precision required for certainty over the exact nature of the errors located within each category. Moreover, in attempting to generate their own categories, the researchers found a lack of literature presenting detailed procedures for the development of error categories. Therefore in addition to identifying the errors students make in simplifying a rational expression, this paper has an additional aim of demonstrating a transparent and easily replicated procedure for the generation of error categories.

The remainder of the paper is organised as follows. The contribution of the work reported here to the field is established through a review of literature on existing methods for categorising errors, predominantly focussing on errors made in the

simplification of a rational expression. Following this, we present a mainly inductive process to categorising the errors made by 95 tertiary preparatory mathematics students when simplifying a rational expression question. This approach resulted in three distinct main error categories. Finally we discuss the results in the context of the existing literature and propose areas for further work.

Literature Review

The literature reveals there are varying approaches to the categorising of errors students make when performing algebra. In general, a common approach to collecting data is to use tests [2-5]. Other approaches include student interviews, student reflections [6] and observations [7]. Sometimes more than one type of data is collected.

Of the literature reviewed, it would appear that approaches to error categorising in algebra are dominated by an inductive approach [2, 5, 6]. For example Storer [2] analysed incorrect solutions to a 52 question test on algebraic fractions to produce 15 error categories, each labelled with a theme that describes features or inferred causes of the errors. Poon and Leung [5] presented twenty-one error categories resulting from the statistical analysis of students' work in an algebra test as well as input from selected teachers. Again, the categories consisted of themes that describe features or inferred causes of the errors in the incorrect solutions analysed. One exception to the inductive approach is the work of Payne and Squibb [4] that used 23 pre-existing error categories from Sleeman [3] and three pre-existing error categories from Matz [8], before defining new error categories to take account of the remaining data collected in their study. Payne and Squibb [4], derived error categories by examining student working on a test of 56 questions, all requiring the solution of a linear equation in one unknown. Error categories were defined with as few as a single occurrence from the working of 86 students across all questions. This is in stark contrast to the pre-existing error categories they used from the work of Sleeman [3] which were defined only if consistent behaviour leading to the same incorrect operations were observed across a sequence of similar questions.

In general, the most common form of output from error analysis is a list of categories, frequencies and examples illustrating each category (see for example, [2-5]). This kind of research informs teaching practice and furthers understanding of student thinking. The study of Storer [2] cites discussions on teaching practice as a motivator for the categorising of errors. The list of errors produced, along with an analysis of their frequency in the solutions students provided was intended to contribute to such discussions. Some studies explicitly focus attention on the use of error categories for understanding student thinking. In these cases, the lists of categories are often of secondary concern and causal links explaining the thinking leading to the observed errors are proposed [3, 4, 6, 7]. Additional data are sometimes used to investigate the causal link, for example Sleeman [3] used interviews. In other cases, the researcher may infer the causes for the errors [7].

Two common weaknesses emerge from reviewing the literature on error categories. Firstly, the process through which the error categories arise is often unclear. Error categories are presented without a detailed procedure that allows subsequent researchers to replicate the study. For example in the work of Storer [2] and Poon and Leung [5], there is no detail regarding how the error categories were constructed, beyond the implied thinking of the researchers that the labeling of the categories suggests. Other studies, such as the work undertaken by Carry, Lewis and Bernard [6], appear overly complicated making them difficult to understand or

replicate. One exception to this is the work of Sleeman [3], where it is possible to reconstruct the study and arrive at a set of error categories that is precisely described.

Secondly, there is often a lack of clarity with how error categories are described. This is caused by authors providing short descriptions of errors usually with some examples, rather than precise definitions (see for example, Storer [2] and Poon and Leung [5]). This lack of precision perhaps explains researchers' lack of referencing to previously defined errors. Lack of precision may also lead to ambiguity.

Such ambiguity is present in literature that categorises errors in the simplification of rational expressions or solution of equations involving rational expressions. The term "cancellation" is frequently used as a descriptor for these errors. Matz [7] describes "cancellation errors" as having the form $\frac{AX+BY}{X+Y} \Rightarrow A+B$. She states (p. 118) for some problems, "the way partial answers are composed into a final answer (superficially) appears more ad hoc. Sometimes signs (particularly minus signs) are ignored, slashed out literals are variously treated as 0 or 1, and not all partial answers always figure in the final answer". Barnard [8] describes two quite different examples, $\frac{x+y}{y} \Rightarrow x$ and $\frac{x+y}{y} \Rightarrow x+1$ as "inappropriate cancelling". Similarly, Parish and Ludwig [9] list examples of errors that are described as "cancellation", such as when $\frac{x+2}{2} = 3$ is incorrectly simplified to $x = 3$. These authors note that they prefer to call this process "obliteration". Poon and Leung [5] categorise the error $\frac{4-12y}{4} \Rightarrow 1-12y$ as: "Misunderstand[ing] the operation of algebraic fractions".

In all of these examples, it appears likely that some form of cancellation occurs; however, the exact mechanics of the cancellation process is unclear. In some cases the most likely inference is that cancellation involves forming a quotient from parts of an expression, while in others, it appears as though the cancellation involves subtraction between quantities on the numerator and the denominator. With the exception of Matz [7], the error descriptions in the literature cited above do not adequately provide other researchers with error categories errors that are unambiguous. We conclude that desirable error categorising protocols should provide categories that have precise definitions and that the method by which the categories have been determined should be detailed enough to be replicated by other researchers.

Method

The research reported here is part of a larger study [10] investigating student learning, in particular, student errors, in the algebraic component of an undergraduate preparatory mathematics course at an Australian university. The one semester course is equivalent to the secondary school mathematics course that prepares students for entry into disciplines such as engineering or the natural sciences where knowledge of calculus is required. A range of students enrol in the course; some have not satisfied mathematics prerequisites for entry to the degree of their choice, while others are enrolled in degree programs that have no mathematics prerequisite for entry but are required to study this level of mathematics during their degree. It is assumed that students enrolled in this course do not have any prior algebraic knowledge.

Data collection instrument

Students sat for the algebra test of 20 questions after having completed the five week long algebra component which comprised approximately the middle third of the course. The test, taken under formal exam conditions, was worth 15% of the total

assessment. Students were directed to show all their working for each question attempted. They were also asked to indicate their level of confidence on a five point scale for each question. In compliance with ethics requirements, the analysis of the data took place in the semester following the delivery of the course. Of the 160 students enrolled in the course in 2010, 151 students had volunteered for the study and of these, 133 sat the test.

The analysis reported here is of the incorrect solutions to the question requiring the simplification of a rational expression. The task required students to

Simplify the following rational expression completely:

$$\frac{b^3 + 6b}{3b}$$

The standard form solution the students had been taught was to factorise the numerator and then cancel common factors from the numerator and denominator:

$$\frac{b^3 + 6b}{3b} = \frac{b(b^2 + 6)}{3b} = \frac{\cancel{b}(b^2 + 6)}{3\cancel{b}} = \frac{b^2 + 6}{3}$$

For this question, 113 of the 133 students (85%) provided a solution with at least one error.

Confidence/memory indicator

To ensure the quality of the data, these 113 responses were filtered using student confidence ratings in an attempt to remove solutions that involved guesswork. The responses from students who used the confidence/memory indicator to indicate that they had “forgotten how to do this type of question altogether” (10) or “didn’t remember seeing this type of question before” (2), were removed. It was assumed that these responses would contain a significant level of guesswork. Similarly, the responses of students who had left the confidence/memory indicator blank (6) were removed, as it was not possible to determine if their working involved guesswork. Using this filtering process, 18 of the 113 responses were excluded from the data set. This left 95 incorrect solutions for coding in which students had indicated that they had been “confident that they were right” (16), “fairly confident that they were right” (49) or “had forgotten how to do bits of this type of question” (30).

Procedure used for coding students’ algebraic working

The process used to analyse the solutions produced a hierarchical coding structure with the “core codes”, at the top level of the hierarchy. Stepwise, the procedure was as follows:

Step 1: The starting point in coding each response was to identify the first process the student appeared to use in their solution. This first step produced two core codes, namely, *Attempted to simplify without factorising* (80), *Attempted to factorise* (15). The responses in each of these core codes were then coded further.

Step 2: The second and subsequent steps in the analytic process involved systematic coding of the working for the solutions in each of the core codes. The coding process involved coding every element of the solution in the order in which it appeared until all the working in the solution was exhausted. An “element” may refer to an operator, a term or a factor of a term. The process involved producing error descriptions in terms of the following three dimensions:

- a) Each element was coded as *correct* or *incorrect* within the context of the prior working relating to that element.
- b) When it was incorrect, inferences were made and recorded concerning the

location from which the error appeared to have emanated.

- c) Each error was also described in terms of the *operations* that the student appeared to have taken to arrive at that particular error.

Step 3: The third step produced the error categories. Using content analysis, the error descriptions were further coded, with a focus on the operation that the student appeared to have performed that led to the error. Error categories emerged from grouping the errors which demonstrated similar operations. Where the data yielded 10 or more individual occurrences of an error, the researchers named the error category and developed a detailed description of it.

Results

The output from Step 1 and Step 2 is presented in Tables 1-4 (see Appendix). These steps led to a total of 26 error descriptions. Common amongst the errors recorded are the mathematical operations of ratio, difference, ratio of like terms but retaining the variable⁶ and factorising. Also evident were errors appearing to involve the interpretation of a sum of two terms as their product and the interpretation of an index as a coefficient.

The grouping of the errors performed in Step 3 is shown in Table 5. To illustrate this process note that the 48 pieces of student working that is recorded in the fourth column of Table 5 is composed of the errors from the second, fourth and fifth columns of Table 1, the second and eighth columns of Table 2 and the third, ninth and tenth columns of Table 4. All of these errors involved forming a quotient in error.

The grouping process led to the identification of six different error categories. In Table 5 these errors are briefly described in the fifth column and named in the sixth column. The description of the three main error categories (where the data yielded 10 or more individual occurrences of an error) is presented in detail below:

The *simple cancellation error* occurs when a rational expression is incorrectly simplified by cancelling a factor that is common to at least one of the terms on each of the numerator and denominator, but that is also not common to all terms on the numerator and denominator.

The *cancellation by subtraction error* occurs when a rational expression is incorrectly simplified in such a way that the resulting expression appears to involve the difference between like terms or coefficients on the numerator and the denominator.

The *cancellation by division of coefficients retaining the variable error* occurs when a rational expression is incorrectly simplified using the ratio of coefficients of two like terms, one on the numerator and one on the denominator where the resultant term is the ratio of the coefficients multiplied by the common variable.

The most commonly observed error was the *simple cancellation error*, (with 48 errors, or 47% of all inferable errors being coded in this category). More than twice the number of errors was recorded in this category than in any other individual category. The other two most frequently recorded errors were the cancellation by subtraction error (22, 21%) and the cancellation by division of coefficients retaining the variable error (20, 18%).

⁶ An example of *ratio of like terms retaining the variable* is so the quotient is processed properly for the coefficients, but not the variable.

Table 5. Coding of Errors Into Categories

Table and location within	Error description	# of errors	Total	Brief error description	Error Name
Table 1, column 2, row 5	The b^2 terms appears to result from $\frac{b^2}{b}$. The $2b$ term appears to result from $\frac{6b}{3}$.	2	48	Resultant term appears as a ratio of circled components.	Simple cancellation error
Table 1, column 4, row 5	The integer 2 appears to result from $\frac{6b}{3b}$.	20			
Table 1, column 5, row 5	The coefficient of $2b$ appears to result from $\frac{6}{3}$.	15			
Table 2, column 2, row 6	The integer 2 appears to result from $\frac{2b}{b}$.	3			
Table 2, column 8, row 6	The b^2 term appears to result from $\frac{b^2}{b}$.	1			
Table 4, column 3, row 17	The term $2b$ appears to result from $\frac{6b}{3}$.	1			
Table 4, column 9, row 11	The integer 2 appears to result from $\frac{6}{3}$.	1			
Table 4, column 10, row 17	The integer 2 appears to result from $\frac{6}{3}$.	5			
Table 1, column 6, row 5	The $3b$ appears to result from $6b - 3b$.	10	22	The resultant term or coefficient appears as the difference between circled components.	Cancellation by subtraction error
Table 1, column 7, row 5	The coefficient of the $3b$ appears to result from $6 - 3$.	3			
Table 2, column 3, row 6	The term b appears to result from $2b - b$.	3			
Table 2, column 7, row 6	The $2b$ term appears to result from $3b - b$.	1			
Table 2, column 9, row 12	The $5b$ appears to result from $6b - b$.	1			
Table 4, column 6, row 11	The $3b$ appears to result from $6b - 3b$.	1			
Table 4, column 8, row 11	The $2b$ term appears to result from $3b - b$.	2			
Table 4, column 11, row 17	The integer 3 appears to result from $6 - 3$.	1			
Table 1, column 3, row 5	The coefficient of the $2b$ term appears to result from $\frac{6}{3}$. The variable b is retained	19	20	Resultant term appears with a coefficient that is a ratio of the circled coefficients, while the variable b is retained .	Cancellation by division of coefficients while retaining the variable error
Table 3, column 6, row 6	The coefficient of the $3b$ term appears to result from $\frac{9}{3}$. The variable b is retained	1			
Table 1, column 8, row 5	The $9b$ appears to result from $3b + 6b$.	2	3	The resultant numerator appears as if the b^3 has changed form to $3b$ then added to another term.	Changing form
Table 2, column 9, row 6	The numerator appears to result from $3b + 3b$.	1			
Table 1, column 9, row 5	The $6b^4$ appears to result from $b^3 \times 6b$.	1	6	The resultant term appears as a product of the circled components	Conjoining
Table 2, column 5, row 6	The $2b^4$ appears to result from $b^3 \times 2b$.	2			
Table 2, column 3, row 12	The b^4 appears to result from $b^3 \times b$.	1			
Table 2, column 10, row 6	The result appears as $2 \times \frac{b^2}{3b}$. Their result may have been treated as a mixed fraction.	1			
Table 3, column 3, row 6	The $2b^3$ appears to result from $b^3 \times 2$.	1			
Table 4, column 2-5, row 5	One term of the numerator is factorised incorrectly	4	4	Only one term is factorised correctly.	Factorising error

Discussion and Conclusions

Overall, students made one of two key choices (the core codes) in attempting to answer the question in this study. Most incorrect solutions (over 80%) resulted from choosing to simplify the rational expression without appreciating that such simplification entailed identifying and then cancelling factors common to all terms in the numerator and denominator.

The approach used to categorise the errors generated descriptions of the algebraic processes that appeared to result in the errors. These were then clustered into categories according to the main operation evident in the error descriptions. The error categories generated give a detailed picture of the difficulties students experienced when simplifying this rational expression.. Valuable information would have been lost if the student working had been categorised using the more general error categories outlined in the literature review. For example, these students' errors could have been categorised using Poon and Leong's [5] "Misunderstand[ing] the operation of algebraic fractions", Barnard's [8] "inappropriate cancelling" or Matz's [7] description of a "cancellation error". In so doing, only a limited indication of the underlying mechanism behind the error would have been apparent.

In this study, the most common underlying mechanisms behind the errors involved incorrect ratios; incorrect differences; and forming ratios of like terms, but retaining the variable. This resulted in the three main error categories presented in the previous section.

Errors similar to the *simple cancellation error* and the *cancellation by subtraction error* can be found in the literature. For example, Carry, Lewis and Bernard [6] describe similar types of errors as "operator errors". In a list of 37 examples (page 43-45), they describe these errors as "a collection of errors which involve the deletion of elements from expressions". The authors explain that these errors appear as approximations of the valid operations of "subtraction from both sides of an equation, division of both sides, division of quotients and subtraction". Here the authors do subcategorise these errors. Of particular interest, 22 of these errors are subcategorised as errors in the "simplification of quotients", while another three are subcategorised as involving "subtracting s[a sub expression] from terms containing it". At first glance the names of these subcategories suggest possible correspondence with the categories *simple cancellation error* and *cancellation by subtraction error* that have been defined in this study. However, on closer inspection it is observed that there is considerable breadth in the categories defined by Carry, Lewis and Bernard [6]. This is almost certainly a consequence of the breadth of their study. The trade-off between depth and breadth in the definitions of error categories needs careful thought, and needs to be considered in the context of the intended outcomes of the error analysis.

The researchers are unaware of any publication describing an error of the form of *cancellation by division of coefficients retaining the variable* error. This is an area for further research. In particular it would useful to see if this result is repeated in similar questions with other cohorts and if so, to investigate student thinking leading to this error.

In conclusion, the analysis of the errors across the whole data set produced three error categories that were defined precisely. The results provide teachers with a snapshot of the difficulties students have in simplifying a rational expression of the type selected. The findings, however, are limited to the simplification of only one example of a rational expression by only one cohort of tertiary preparatory mathematics students. Further research is required with other forms of rational

expressions and with more students. More research is also required to test the applicability of the error analysis process documented and illustrated here to the analysis of student errors in solutions to a broader range of problems.

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Appendix

Table 1. Description and Frequency of First Set of Errors in Core Category “Attempted to Simplify Without Factorising”

Simplifying	Incorrect						
Total No.	72						
Location of errors	$\frac{b^3+6b}{3b}$ & $\frac{b^3+6b}{3b}$						
Errors	$\frac{b^2+2b}{b}$	b^3+2b	b^3+2	$\frac{b^3+2b}{b}$	b^3+3b	$\frac{b^3+3b}{b}$	$\frac{b^3+6b}{3b}$
Description of errors	The b^2 terms appears to result from $\frac{b^2}{b}$. The $2b$ term appears to result from $\frac{6b}{3b}$.	The coefficient of the $2b$ term appears to result from $\frac{6}{3}$. The variable b is retained.	The integer 2 appears to result from $\frac{6b}{3b}$.	The coefficient of $2b$ appears to result from $\frac{6}{3}$.	The $3b$ appears to result from $6b - 3b$.	The coefficient of the $3b$ appears to result from $6 - 3$.	The $6b^4$ appears to result from $b^3 \times 6b$.
Error count	2	19	20	15	10	3	2
# students continuing	0	1 ** Go to Table 3	1 **** Go to Table 3	10 * Go to Table 2	1 *** Go to Table 3	3 **** Go to Table 2	1 **** Go to Table 3
							1 0

Simplifying	Incorrect						
Total No.	7						
Location of errors	$\frac{b^3+6b}{3b}$						
Errors	$\frac{b^3+6b}{3b}$	$\frac{6b^3}{3b}$	$\frac{3b^4}{b}$	$\frac{b^3}{2}$	$\frac{b^3+6}{3}$	$\frac{2bb^3}{3b}$	$\frac{b^3+6b}{3b}$
Description of errors	Not inferred	Not inferred	Not inferred	Not inferred	Not inferred	Not inferred	Not inferred
Error count	1	1	1	1	1	1	1
# solutions continuing	1 **** Go to Table 3	1 **** Go to Table 3	1 **** Go to Table 3	1 **** Go to Table 3	1 **** Go to Table 3	1 **** Go to Table 3	1 **** Go to Table 2
							0 1 **** Go to Table 2

Table 2. Coding of Further Errors in Core Category “Attempted to Simplify Without Factorising” (continued from Table 1)

Starting point	*										****
Simplifying	Incorrect										***
Total No.	10										Incorrect
Location of errors											1
Errors											3
Description of errors	The integer 2 appears to result from $\frac{2b}{b}$.	$\frac{b^3 + 2b}{b}$	$\frac{b^3 + 2b}{b}$	$\frac{b^3 + 2b}{b}$	$\frac{b^3 + 2b}{b}$	$\frac{b^3 + 2b}{b}$	$\frac{b^3 + 3b}{b}$	$\frac{b^3 + 3b}{b}$	$\frac{b^3 + 3b}{b}$	$\frac{b^3 + 3b}{b}$	The result appears as $2 \times \frac{b^3}{3b}$. Their result may have been treated as a mixed fraction.
		$b^3 + 2$	$b^3 + b$	Not inferred	The $2b^4$ appears to result from $b^3 \times 2b$.	Not inferred	$\frac{2b^4}{b}$	$\frac{2b^3}{b}$	The b^2 term appears to result from $\frac{b^3}{b}$.	The numerator appears to result from $3b + 3b$.	
Error count	3	3	3	1	2	1	1	1	1	1	1
Simplifying # solutions continuing	N/A	Incorrect	Incorrect	N/A	Incorrect	N/A	N/A	N/A	Incorrect	N/A	N/A
Location of errors	N/A	1	1	0	1	0	0	0	1	0	0
Errors	N/A	$b^3 + b$	$b^3 + b$	N/A	$\frac{2b^4}{b}$	N/A	N/A	N/A	N/A	$\frac{b^3 + b + b + b + b + b}{b}$	N/A
Description of errors	N/A	N/A	b^4	N/A	$2b$	N/A	N/A	N/A	N/A	$5b$	N/A
			The b^4 appears to result from $b^3 \times b$.	N/A	Not inferred	N/A	N/A	N/A	N/A	The $5b$ appears to result from $6b - b$.	N/A

Table 3. Coding of Further Errors in Core Category “Attempted to Simplify Without Factorising” (continued from Table 1)

Starting point	**	***	**	****	*****	*****	*****	*****	*****
Simplifying	Incorrect	Incorrect	Incorrect	Incorrect	Incorrect	Incorrect	Incorrect	Correct	*****
Total No.	1	1	1	2	1	1	1	1	*****
Location of errors	$b^3 + 2b$	$b^3 + 2$	$b^3 + 3b$	$\frac{9b}{3b}$	$\frac{6b^3}{3b}$	$\frac{b^3}{2}$	$\frac{b^3 + 6}{3}$	N/A	$\frac{b^3 + 6b + 6b + 6b + 6b + 6b}{3b + 3b - 3b}$
Errors	$2b^3$	$2b^3$	$\frac{3b}{b-3}$	$27b$	$\frac{2b^2}{1b}$	$\frac{b}{2-3}$	$\frac{b^3}{2}$	N/A	$\frac{b^3}{9b}$
Description of errors	Not inferred	The $2b^3$ appears to result from $b^3 \times 2$.	Not inferred	Not inferred	Not inferred	Not inferred	Not inferred	N/A	Not inferred
Error count	1	1	1	1	1	1	1	1	1
Simplifying	N/A	N/A	N/A	N/A	N/A	Incorrect	Incorrect	N/A	N/A
# solutions continuing	0	0	0	0	0	1	1	0	0
Location of errors	N/A	N/A	N/A	N/A	N/A	$\frac{b}{2-3}$	$\frac{b^3}{2}$	N/A	N/A
Errors	N/A	N/A	N/A	N/A	N/A	$\frac{b}{-8}$	$\frac{b}{2}$	N/A	N/A
Description of errors	N/A	N/A	N/A	N/A	N/A	Not inferred	Not inferred	N/A	N/A

Table 4. Description and Frequency of Errors in Core Category “Attempted to Factorise”

Factorising	Incorrect						Correct	
	6						3	6
Total No.								
Location of errors	$\frac{b^3 + 6b}{3b}$						N/A	N/A
Errors	$\frac{3b(b^2 + 2)}{3b}$	$\frac{b(b^2 + 6b)}{3b}$	$\frac{6(b^2 + b)}{3b}$	$\frac{b^2(b + 6)}{3b}$	$\frac{b^3 + 6b}{3b}$	$\frac{6(b^3 + 1)}{3}$	N/A	N/A
Description of errors	One term of the numerator is factorised incorrectly						N/A	N/A
Error count	1	1	1	1	1	Not inferred	N/A	N/A
Simplifying	Correct	Correct	Incorrect	N/A	Correct	Correct	Incorrect	Correct
# solutions continuing	1	1	1	0	1	1	2	5
Location of errors	N/A	N/A	$\frac{6(b^2 + b)}{3b}$	N/A	$\frac{6b(b^2)}{3b}$	N/A	$\frac{b(b^2 + 6)}{3b}$	N/A
Errors	N/A	N/A	$\frac{2b^3}{3}$	N/A	$\frac{3b(b^2)}{3b}$	N/A	$\frac{b^2 + 6}{2b}$	N/A
Description of errors	N/A	N/A	Not inferred	N/A	The 3b term appears to result from $6b - 3b$.	N/A	The 2b term appears to result from $3b - b$.	N/A
Error count	N/A	N/A	1	N/A	1	N/A	2	1
Simplifying	N/A	Incorrect	Incorrect	N/A	N/A	Correct	N/A	Incorrect
# solutions continuing	0	1	1	N/A	0	1	0	5
Location of errors	N/A	$\frac{b^2 + 6b}{3}$	$\frac{2b^3}{3}$	N/A	N/A	N/A	N/A	$\frac{b^2 + 6}{3}$
Errors	N/A	$b^2 + 2b$	2	N/A	N/A	N/A	N/A	$b^2 + 2$
Description of errors	N/A	The term $2b$ appears to result from $\frac{6b}{3}$.	Not inferred	N/A	N/A	N/A	N/A	The integer 2 appears to result from $\frac{6}{3}$.
								The integer 3 appears to result from $6 - 3$.

Teaching Takes Place *In* Time, While Learning Takes Place *Over* Time: Maturation of Mathematicians Seen as Phase Transitions in Te Ara Mokoroa (The Long Abiding Path of Knowledge)

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Significant states before and after ‘transition’ from secondary to tertiary study of mathematics have been clearly identified ([1], [2]). Arising from reflections of a tutor who herself experienced a transition between countries as an undergraduate and then a graduate, and then to tutoring, we suggest that although teaching consists of a sequence of acts in time, learning is most appropriately seen as a maturation process that takes place over time the major part of which is not actually visible.

Key Words: Transition, undergraduate, tutoring, maturation, phase transition, gargling, babbling

Background

As a phase of eight years of short-term contracts tutoring mathematics comes to an end for the first author (G), reflection on her experience, stimulated by an interview with the second author, gives rise to insights into the experience of transition for herself, and through this, the experience of her students, admitted to a prestigious UK university.

Transition from high school to university

In seeking a classification of perspectives on and experiences of transition, De Guzman *et al.* [3] distinguish between epistemological, cognitive, socio-cultural and didactical perspectives. In a comprehensive review of the literature Gueudet [2] re-organizes these perspectives, moving her focus from questioning individual behaviours to more socio-cultural and eventually institutional influences. She notes that “difficulties observed, didactical action, and view of transition are certainly intertwined, and they also modify the *a priori* idea of difficulties” (p239), making the study of transitions complex.

In [1], Tall suggested that “The move from elementary to AMT [advanced mathematical thinking] involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on definitions” (p20). The stimuli offered in this paper provide specific and detailed examples of a tutor’s recollections of supporting and encouraging this transition, seen as a maturation process over time. The contribution being offered is a detailed description with comments using labels for psychological and pedagogical constructs, that could stimulate others to reflect similarly on their own practices.

Some students concentrate on picking up phrases and behaviours and then attempting to use these as if in a formal game [4]. We use the term *gargling* to describe assembling reasoning out of fragments of formal language in the hope that meaning will emerge, or at least that the tutor will be convinced [5], and we

distinguish this from *babbling* (first used in [6] for secondary students), which describes significant attempts to express some relationship or property even though it is at best confused and at worst scrambled mathematics. Berger noted similar behaviour at tertiary level [7].

Phase transition

Despite being exposed to concepts repeatedly and appearing to ‘understand’, many students act as if they have never before met the idea. For example, Helen Drury (2006 personal communication) encountered a student who in several lessons gave the distinct impression of experiencing a fresh epiphany each time concerning the effect of multiplication by zero. One way to account for this phenomenon is by analogy to phase transition in physics in which re-introduction or correction (energy) is apparently ignored (no significant temperature change) and then suddenly a change takes place in student behaviour (temperature rises again). It seems that maturation takes time (and energy) for robust state transition(s) to take place.

Approach

Our approach is entirely pragmatic and opportunistic in the territory common to phenomenology [8] and the Discipline of Noticing [9]. We are interested in the lived experience of students and teachers and in developing ways of sensitising ourselves, and others, to significant issues, with associated alternative actions that might be initiated.

Our starting point is G’s reflections on her experience arising from a 90 minute interview. However, our approach treats her reflections not as data, but as stimuli for seeking resonance and dissonance with the experience of colleagues. Thus our actual data consists of the memories and experiences that come to the surface for readers who encounter what we present. Some of these memories and experiences may then crystallise as shared phenomena. Our aim is to offer readers insight into their own experience of teaching. We do this by presenting G’s reflections on encountering and learning to work within an unfamiliar system of lectures and tutorials, and through her developing sensitivities, arising from her attempts to gain insight into the experience of students as they matured into mathematical thinkers. Thus G is ‘learning over time’, and, we submit, each tutor has to go through a similar process over time. There is no quick way to developing one’s teaching. Our aim is to enrich the field of experience of readers rather than to make ‘objective’ or ‘factual’ claims.

Setting

The tutor meets the first year students weekly for each 8 week term and at least as frequently in the second year. The lecturers in the mathematics department set problems and originate all assessment that contributes to the students’ final grade. Despite this department-centric structure, the tutor is seen as the principal ‘teacher’. In the first year the tutor usually meets pairs of students, with students handing in assignments prior to the tutorial so the tutor can see their work beforehand. However, alternative arrangements exist, such as having a weekly class during which the assignment questions are reviewed with the tutor providing model solutions, while tutorials in pairs take place on a fortnightly basis. Students need not hand in any work for these latter.

A tutor has to provide an environment in which students can work and engage with the topic. How a tutor does this is his/her choice, but it is ‘understood’ that a tutor

might ask questions to the students on theory, on their answers to set problems and on unseen problems, and that the tutor might decide to discuss a topic not strictly on the syllabus.

Stimuli

Italicised questions are the questions asked at interview. Indented paragraphs are edited extracts from the interview. The remainder of the text is commentary from us both.

Influences on tutoring

What was your initial experience of tutorials?

During my first two years I did essentially the same tutorial four times, repetitively. I addressed the same issues in each tutorial with their work in front of me. My sense was that they had given me what they had to give; the tutorial was my turn to deliver.

A didactic contract is clearly in operation, even if the terms have not been agreed explicitly. Students act as though their sole task is to solve set problems.

After two years I changed college and encountered a different system: there was a problem class (all students together, essentially to find out how they could have answered the set problems and in good years you get much more serious interaction) and a fortnightly tutorial (no longer a pre-set agenda ... no problem sheet). What to do for an hour with pairs of students? There was more flexibility so I was able to respond to students.

... Responding to students?

... students reacted in different ways ... some might not have needed what I chose to rehearse; it was often boring for me to do the same thing four times over! Timing ... sometimes I finished early, sometimes I only covered half the material.

Different timings and different emotional evaluations of herself and of student behaviour suggested that G was responding partly to her students and partly to her own psychological state. She became explicitly aware of choices to be made: how to spend the time most profitably from the students' point of view. It took time to learn to listen (in a broad sense) to her students more than to her own state. It also took time to recognise that students were trying to express their thinking but not always succeeding. For example

a student for whom English was a second or third language, having met *bounded above* and *bounded below* asked me "what is bounded?"

I had never considered this to be problematic, but it gave me a taste of what it might be like for students in terms of how they think about technical terms.

Incidents like these amplified G's desire to promote more active engagement in tutorials, and to watch out for unexpected overlap between technical and every-day use of terms.

That is one of the things that made me be more careful about this ... talking all the time, especially looking at their faces more to see more from their expressions whether they were lost, whether I had lost them, whether we were talking about the same thing, whether they were interested in what I was talking about ... sometimes they get into 'yes' mode but a bit blank and then I try to see if we can do something else. ... trying to stop myself from covering the silence by using my own voice. I tend to do it in conversation with other people so it is easy to fall into that trap. I

listen much more [now]. My personal propensity is to fill silence, in tutorials and outside, while the students remain quiet until certain.

When students assent without attempting to express ideas for themselves, it is tempting to assume that meanings are shared, when in fact they are at best 'taken as shared'. If students have little or no opportunity to try to express themselves, to assert, then they are hampered from gaining control over and integrating the concepts into their functioning [10]. Getting students to move from assenting to asserting, from waiting to conjecturing requires explicit action on the part of the tutor.

The impulse to talk to the students in tutorials arose from a collection of mutually amplifying forces: personal propensity, the problems-tutorial institution, and the reluctance of students to display ignorance. There is of course nothing new in this self-recognition: every tutor-teacher has to get to grips with the tension between the impulse to talk and the value of listening deeply [11]. The roles of 'speaker' and 'listener' become part of the didactical contract [12] between tutor and students. Breaking implicit conventions and expectations based on past experience might be difficult and is sometimes resisted at first by students.

Student reticence

Finding students reluctant to contribute in sessions is a common phenomenon. G begins by referring to an unusual case of a student who was able to talk publicly about what he had done, whereas many if not most are exceedingly reluctant:

Unusually, one second year student was able to tell me what he had done in a problem ... because when the question becomes quite technical, I find that even the best students find it difficult to say aloud what they have done. There are some exceptions. ... but for a lot of them, even if I ask them to copy it on the board ... still they don't seem to write on the board what they have on their paper. I don't know how this happens. But it's very difficult.

G then goes on to offer a complex of factors that could be operating:

They find it hard to put their mathematical ideas out for discussion. ... Even in the tutorial system many mathematicians are of introvert variety, partly insecure, partly because of this new environment. Perhaps they think that everyone around them is much brighter than they are, they don't want to put their ideas on the table and have them discussed.

This matches informal responses from students who often express reluctance to expose their ignorance at any time, and especially in the presence of peers who seem to be much cleverer [13].

In the first few tutorials you tend to see those who are very sure about what they know, they'll answer things, and those who are insecure will not even [try] though they possibly know the answer.

So that is another way I am listening, not listening specifically to what they say but listening to their actions and body language.

Becoming aware of the cues that are informing choices is an important step towards improving effectiveness for students.

How has your handling of these initial tutorials changed over the years?

In the first years I followed the problem sheet ... and I wasn't really thinking about, for example, whether I wanted to link any questions, while in the last couple of years sometimes I haven't used them at all ... I have given them the problems and asked if there was anything of interest here or if I had the impression that they could do most

of them then perhaps we look at other things. For example ... injective, surjective and things which are close to what they are doing in the first weeks because I know from previous years that they struggle with those [concepts]. I would rather get them to practice [using them]: can you write down an injective function, and another one, can you find one that is injective but not surjective We do example construction.

Perhaps other tutors use these initial questions in a different way ... I find them limiting or I can't think of ways of moving out of the boundaries of that question. In some cases I just do something else instead.

G describes what must be a common phenomenon of student reluctance to contribute, especially on entrance to university. Over time she developed strategies, focusing on concepts that students always struggle with, and engaging them in mathematical activity of constructing examples for themselves, an important study technique for them to internalise [14].

Shifts in behaviour

Learning 'the game'

The necessity to justify actions, whether in a proof or in the use of a theorem is made explicit at every lecture and tutorial. Even so, it takes some students considerable time to get to grips with what is required.

The main issue for many students in the first term is that [they act as if] they think that the point is the final answer rather than the whole process by which you arrive at the answer. In some cases they learn conventions very quickly, and in some cases after two or three terms they still focus on answers.

The first term works so hard at getting them to prove everything and anything. By the time they reach the second term ... they think they know what continuity is, they think they know what differentiability is. But by that time most of them have learned that when they think they know, it doesn't mean they can do it the way they used to do it at school and so they tend to be more cautious.

This accommodating to the new mathematical culture must contribute to amplifying reluctance to perform in front of others. This is more than a socio-cultural mathematical norm in the sense of Cobb [15] or Rasmussen [16]. A recent student, encountered after a period of G being on leave, was considered by other tutors to be weak:

"She's very weak ... she's not justifying anything." "She doesn't understand anything". But I have seen her in tutorials and though she's not the strongest student, she's better than the feedback from people who were marking her work [She] hasn't quite learned what is wanted, what is expected in an answer. Even with a technique such as the computation of a determinant, the marker expects to see the statement that the determinant is to be calculated via ... We will see after Easter if the message got through ... if she is not justifying her work even though we told her three or four times, then we'll have to see if [her problem] is mathematical and she doesn't [understand] ...

It turned out that after the break between terms she did well in her termly examination and her work the following term was much more articulate and well justified. This is an example of maturation taking place *over* time, even though teaching takes place *in* time. It behoves the tutor to be aware of this as a common phenomenon and not to expect immediate changes in behaviour.

Students maturing from first to second year

How are students different in second year?

Do you have an example of maturation taking place over a short period?

... the formal repetition of epsilon-delta proofs (always produced along ritualised lines, including the choice of words “Let epsilon greater than 0 be given ...”) appears to give students a reassuring environment, each repetition playing some role in providing new meaning. I was amazed to see this work, as I always assumed that students should be encouraged to understand from 'the start'.

Ritualised repetition can lead to meaning, whether in the short term or over the long term.

In the second year most students are confident with the use of epsilon-delta proofs and can produce their own. And it amazes me that it does happen, while I am frustrated that up to a couple of weeks before first year exams many still appear to be lost.

Pressure of imminent examinations is certainly a strong force, but not the only one.

In the second year the number of tutorials with a given tutor can vary as it depends on the student's choice of optional courses, but G sees her students at least weekly (by the time the revision term comes G might be seeing each student 3 to 5 times a week).

For the majority of the students by the time they enter the second year, for example, they know when they don't know how to answer a question ... so there's much less “I'll try to see if I can write something which my tutor might think is the right answer but I have no clue what I'm doing”. You get none of that.

The term *gargling* is a useful label for the latter approach, and distinguished from *babbling* [5] when students are evidently trying to express themselves in technical language but not using appropriate syntax.

[by second year] there's no gargling ... very little. The weekly tutorials contribute to that. There's the occasional student that will try in a special case to do it but as a rule, no ...

Marton & Säljö use the terms *surface* and *deep* to distinguish student approaches to learning [17, 18]. Babbling and gargling are descriptive of students' local behaviour, while surface and deep are descriptions of tutor's interpretations of students' general or global behaviour. Informal remarks by students from other universities suggest that gargling may be a more widespread strategy than amongst those whom G meets.

What are tutorials like in the second year?

In the second year it's extremely different. I prepare for the tutorials, but its very unlikely I will be left with “Oh my goodness what are we going to do?”. In the second year, they have the set problem sheet which they have attempted, but not the problem classes. So the content is already available. ... They are much more aware of what they understand. ... They use the tutorials much more for really understanding and they talk in a pretty articulate way about their answers.

I take them through the problems in order. If I think there are some tricky bits I ask them more specifically “how did you do this bit?”. ... There was a case this year when they said “Oh we all did problem 4 but none of us used the condition that ... it wasn't necessary”. So they thought they had an answer but they were also

sophisticated enough to realise that there might have been a problem [if they hadn't used all the conditions] ... in a first year question you would have some who noticed that they didn't use an hypothesis, but others who wouldn't even notice.

There is an evident change in most students from first to second year ...

... but I will never know whether it is exam preparation that brings them to this or ... dropping them in June and seeing them again in October. In some cases it's hard to believe they are the same people in front of me.

Regarding a particular student during the summer away from the tutor ...

A lot of work over the summer? I don't know when he's done the work but its not the same student. Last year he sort of ... he was putting [in] the minimum effort ... a student who has potential. He achieved very very well at school and perhaps he thought he could catch up later on as he did at school. ... This year his problem sheets have been first class. It's not just that the answers are right but he's justifying; he's explaining what he is doing. Last year perhaps he knew what he was doing but he wasn't articulating what he was doing. Now he is always explaining his reasoning in tutorials as well as on paper.

We see this as another instance of teaching taking place in time (in tutorials) but learning taking place as a maturation process over time, at least partly in response to enculturation into this practice. Continued urgings appeared to have no effect, and then in the second year they bore fruit, rather in the way that phase transitions occur in physics.

Maturing as a tutor

At first G used the problem sheets as the sole source of stimulation around which to structure tutorials, but as she grew into her role she found herself using them in a more flexible way, bringing in other problems when she thought it appropriate.

Gaining experience of what students find difficult, for example struggles with the definitions and use of *surjective* & *injective* functions, leads most tutors to take immediate recovery action and to modify their practice in subsequent years. This can be an *ad hoc* 'empirical' process of trial and improvement, but it can also be informed by growing familiarity with the mathematics education literature. It is one thing to alter a local practice (for example, choosing to focus on these constructs early on rather than waiting for difficulties to emerge in later tutorials), but it is quite another thing to see this as an instance of a more general phenomenon and to adjust tutorial practice accordingly.

G was led to ponder the 'pressure-cooker' pedagogy of the institution. There was ...

no time to look back, so in tutorials you get the phenomenon of not recognising, not having-come-to-mind things met several weeks earlier.

One thing I wish I had realised earlier on. I don't really know how to fit it in. I had increasingly this feeling that learning is so structured, you know, the problem sheets and all of that: it's a double edged sword. At the beginning I really thought it was just very good. It supported students very well and I wished that I had even more of that when I was a student. Now I have come to this feeling that perhaps there is almost too much of it in such a short term: students get to the end of the term and they come up gasping for air and they haven't had any time whatsoever to look back at anything. They have to do all their work when they go away. So I don't see most of their [maturing], it happens when I am not there.

The phenomenon of students not having come to mind things they have been

taught recently re-occurs at every level. If students are not immersed in concepts for sufficient time before they are employed as components of yet further concepts, it is not surprising that things don't come readily to mind.

G's reflections led to an emergent personal pedagogy, in which she tried to use mathematical stimuli to generate mathematical discussion with the potential to lead to connections to recent experience in lectures based on past experience of student struggles. This again is an example of trying to establish practices as socio-cultural mathematical norms [15, 16].

I try to organise one or two sessions every year, either as a special class or as part of a tutorial, during which students can work together on problems that look and read differently to the weekly problem sheets. At first I was responding to student difficulties but then it became part of my practice. These give students an opportunity to pause, away from the weekly curriculum. It worries me that in the weekly problem sheets they know which topics are covered, so that they are deprived of the need to figure out on their own which results to use.

A corresponding practice emerges over time as G responds to her students, seeking ways to improve their experience.

Reflection

Drawing on De Guzman's distinctions [3] we briefly consider G's observations from the point of view of epistemological, cognitive, socio-cultural and didactical perspectives.

Socio-cultural perspective

Students' reluctance to expose their ignorance is entirely natural, but unhelpful if mathematical thinking is to develop and mature. An atmosphere needs to be established in which students are willing to make conjectures and to modify those conjectures on the basis of other people's contributions [19]. This is essential not only for effective group work, but also for individual work.

Didactical perspective

Negotiating aspects of the didactic contract explicitly may be of some help to some students: it is ineffective in the long run to achieve tasks using surface strategies [10 p89] rather than learning how to learn mathematics by engaging deeply. Listening deeply to students while being sensitive to the phenomenon of babbling as students try to express themselves is a skill that has to be learned through experience. Getting students asserting conjectures rather than assenting to what they are told can contribute to students taking initiative [10].

Cognitive Perspectives

As Tall observed, the essence of first courses in university mathematics is usually to do with the role of definitions, the nature of mathematical reasoning, and the construction of proofs [1]. This requires students to let go of any addiction they have developed to 'doing calculations to get answers', shifting instead to giving reasons for the appropriateness of those actions and proving structural relationships. Students who do not make this transition either drop out or attempt to get through on the basis of memory and gargling. This is confirmed by other studies of students [e.g. 10 p89].

Epistemological perspectives

One of the issues that comes across to us is the inappropriateness of the pervasive ‘staircase’ metaphor for learning, in which students are set tasks of increasing complexity as they ‘climb’ the staircase of sophistication. It draws upon an explanatory mechanism of cause-and-effect: teachers and students act in certain ways and learning is the result. Such a mechanistic Cartesian world view simply does not fit with experience of education. Based on close observation, Pirie & Kieran [20] built into their ‘onion’ model of the growth of understanding phases of ‘folding back’ as students appeared to re-enter previous states yet informed by recent experience, before consolidating and integrating so as to display behaviour associated with deeper and more comprehensive understanding. G’s observations reported here of the transformation (no mere transition) in the state of so many of her students on the ‘long abiding path of knowledge’ between the end of their first year and the beginning of their second year cannot, in our view, adequately be accounted for through cause-and-effect. Specific acts such as working hard, going over notes, reworking problems, and reading ahead are unlikely in themselves, or even in concert, to bring about the change of perception and stance observed by tutors.

Phase transition is an attractive metaphor for what happens when an intensive 8-week term is followed by releasing the pressure and providing time for assimilation, integration and maturation as seems to happen for many students. Energy is poured in during term and small changes in behaviour are visible; then further effort and no obvious improvement is followed by noticeable change in behaviour. Our own experience bears this out. At some point or other a course suddenly ‘makes sense’ or ‘fits together’, hopefully before the examination takes place!

There are so many active factors that it is not enough to say that ‘working hard’ and then ‘letting go’ will result in an appropriate phase transition. Phase transition is still based in cause-and-effect. An even more fitting metaphor might be found in chemical reactions, where the presence of catalysts and a multitude of agents can (but may not) generate conditions in which multiple phase and state transitions take place. Such a metaphor would preserve the complexity of teaching and learning without reducing it to overly simplistic cause-and-effect mechanisms which simply do not transfer from one situation to another, despite the hopes and desires of policy makers.

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Limits to Growth and Mathematical Basics

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Elementary processes from different branches can be described by ordinary differential equations of first order. First well-known models of growth are studied (linear, exponential and super-exponential growth, logistic growth and generalizations, growth with time-dependent growth rates). The problem of adapting parameterised growth functions to statistical data is mentioned. Further simple models with several steady states are considered. In this context the concept of stability is introduced. Then interesting effects can occur deepening the understanding of the interplay between dynamical changes and steady states. There are a lot of possibilities to influence the behaviour of processes by parameter change. Already simple models can reveal important consequences of real processes and can give hints for intervention. But mathematical basics are needed to understand and to exploit this. Therefore the necessary mathematical pre-knowledge is outlined. Using project work in teams the students are motivated to solve practical problems and to learn more mathematical basics.

Keywords: growth; differential equations, steady states; modelling and simulation; projects and team work

Growth in Nature and Society

Growth is a general phenomenon. In poor countries often the human population grows explosively. In developed countries business hopes to grow strongly for increasing profit. More profit is necessary to survive in the international competition and to invest into new enterprises. The people in a region hope also to profit from the growth of trade and industry. With growing industry also the consumption of material and energy as well as the pollution and destruction of environment increase. To top it also the temperatures on earth, the hole in the ozone layer, the general level of water and the power of wind increase. Nevertheless it seems to be clear that growth is limited earlier or later. Therefore growth is also an emotive word in the political discussion (see [3]).

Mathematics is not very popular although it is hidden behind all things. For this reason growth is investigated from a point of mathematical basics showing the benefit of mathematics and motivating students to learn it. At the same time aspects of modelling and simulation are included demonstrating the interplay of different disciplines for the solution of actual problems. Problems are solved by algorithms on a computer. If the input data have a certain magnitude especially the effort necessary for solution is of interest (computation time, storage capacity). Also here growth comes in since the big problems in science and society need the management of huge data sets. If the solver is clever using effective algorithms he can save a lot of capacities.

Project Work

One idea of modern teaching is breaking up the classical frontal teaching by more

cooperative forms. Another idea is to use modern means as computer for computation and exploration (see e.g. [1]) or internet for information and teaching material. Occasionally projects are introduced at the engineering faculty in Wismar where teams of students work on practically relevant problems with a mathematical background guided by staff members. Mathematical computer software as MATLAB is used as an additional tool. In the past good experiences were made with such topics as computerized tomography (CT), oscillators in engineering, predator-prey models, traveller-salesman problems or strategic games (see e.g. [6], [7]). Also the reactions of students were very positive in general. The level of mathematics can vary on a wide range and has to fit to the pre-knowledge of students. If projects are included in the first year mathematical education a revision of syllabus can be necessary. Another possibility is to offer projects in higher semesters as optional courses. Then integration of engineering subjects and cooperation with colleagues from engineering is fruitful.

The new project about growth processes needs only mathematical basics and calculus. But it can also be extended including such subjects as ordinary differential equations, numerical mathematics, probability and statistics as well as computer mathematics. Since all these subjects occur in our first year course of engineering mathematics at most the order of content has to be changed. Some of the usual topics can be cancelled to avoid overloading of the course.

Growth Functions and Data Fit

In a continuous context growth should be looked for in the class of time functions $x: T \subseteq \mathbb{R} \rightarrow \mathbb{R}$ which are nonnegative and (strictly) monotone increasing. Normally it is supposed that $T = [0, +\infty[$ and $x(0) = x_0 \geq 0$. Further, the functions should be smooth enough to guarantee the application of calculus. The first derivative $x' = x'(t)$ indicates the *velocity* or *rate* of growth, while the second derivative $x'' = x''(t)$ describes its *acceleration*. The quotient

$$r = r(x, t) = \frac{x'(t)}{x(t)}$$

means the *relative growth rate* of x . For growth functions x it is $x(t) \geq 0$ and $x'(t) \geq 0$ (i.e. x is monotone increasing). The growth is called *progressive* for $x''(t) \geq 0$ (i.e. x is convex) and *digressive* for $x''(t) \leq 0$ (i.e. x is concave). Finally, it is differed between unbounded, explosive and bounded growth. Within these classes different scales of growth can be introduced corresponding to the asymptotic behaviour of growth functions.

Growth is called *unbounded*, if it goes beyond all limits in time: $\lim_{t \rightarrow +\infty} x(t) = +\infty$. Growth is said to be *explosive* if it goes beyond all limits already in finite time. Restricting the domain to $T = [0, a[$ this means $\lim_{t \rightarrow a-0} x(t) = +\infty$. Hence x has a *pole* at a .

Growth is *bounded* if $\lim_{t \rightarrow +\infty} x(t) = C$ for some $C > 0$. The constant C is called *capacity*. Then the rate of growth is decreasing to 0: $\lim_{t \rightarrow +\infty} x'(t) = 0$. There are two typical cases, type 1 of digressive bounded growth (without turning point) and type 2 of bounded growth starting progressive and turning to digressive at a certain time $t_w > 0$. In the second case the function curve has a sigmoid form (see figure 1).

Who is familiar with stochastics recognizes an interesting cross connection: all *distribution functions* are examples for bounded growth with capacity 1.

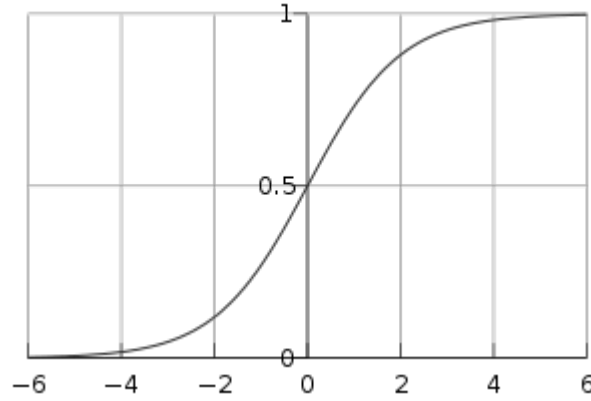


Figure 1. Sigmoid logistic curve.

Counterparts of monotone increasing functions are monotone decreasing functions $x : T \subseteq R \rightarrow R$. In this context such functions should be nonnegative. They describe processes of decline or shrinkage. If x is also bounded, then typically x is convex (type 1) or turning convex from some $t_w > 0$ (type 2).

Now real growth processes are considered. Assume there is a list of data (t_i, x_i) for $i = 1, K, n$ and a family of growth functions $x = x(t; p_1, K, p_m)$, where p_j ($j = 1, K, m$) are parameters ($m \leq n$). The problem is to find the function in the family fitting the given data best. In the case $m = n$ there are n equations for the n unknown parameters. Generally there is a unique solution supplying parameters such that the corresponding x interpolates the given data. Otherwise the method of *least squares* is used, where

$$\sum_{i=1}^n (x(t_i; p_1, K, p_m) - x_i)^2 \rightarrow \text{Min}.$$

By partial differentiation of this expression with respect to the parameters and by putting all m obtained terms to 0 (condition for local extremes) a system of equations occurs which has to be solved for the unknown parameters. This part is not elementary. But software like MATLAB can be used.

Growth Models and Steady States

In the following differential equations $x'(t) = f(t, x(t))$ of first order are studied which describe growth processes $x(t) \geq 0$. Initial conditions $x(0) = x_0 \geq 0$ are given to make the process unique. If time t does not explicitly occur in the differential equation, it is called *autonomous*. Such an equation can be solved by separation of the variables. An explicit resolution for x is only possible if G is invertible:

$$x' = f(x) \Rightarrow G(x) = \int \frac{dx}{f(x)} = \int dt = t + C \Rightarrow x = G^{-1}(t + C).$$

Here C is an arbitrary constant. If x_0 is given, then $C = G(x_0)$. Considering the dynamical behaviour of a growth process *steady states* (equilibriums) are of special interest. Steady states x_s occur for vanishing velocity x' . For an autonomous differential equation $x' = f(x)$ these states are zeros of f : $f(x_s) = 0$. Eventually

numerical methods are necessary to determine these zeros at least approximately. Two different kinds of steady states are possible: stable ones which attract the process and unstable ones which distract it. For a simple analysis it is sufficient to study the process behaviour locally in a small neighbourhood of the steady state. Approximating $f(x)$ around x_s by using the tangent supplies

$$x' = f(x) \approx f(x_s) + f'(x_s) \cdot (x - x_s) = f'(x_s) \cdot (x - x_s).$$

But then $f'(x_s) < 0$ (negative ascent) means that small changes of x away from x_s lead to a return. Hence, x_s is *stable*. On the other hand, if $f'(x_s) > 0$ (positive ascent) then the process is running away from x_s . Hence, x_s is *unstable*. So it is easy to recognize the stability of steady states considering the curve of f and the ascents at the zeros or calculating the derivative.

Linear and exponential growth

If the velocity of growth is a positive constant, then the growth is linear:

$$x'(t) = a \quad (a > 0), \quad x(0) = x_0 \Rightarrow x(t) = a \cdot t + x_0 \rightarrow +\infty.$$

This growth is unbounded and has no steady state. If natural growth occurs developing freely by acting of inner forces without any restrictions, the growth rate is a positive constant. The velocity of x is proportional to x itself:

$$x'(t) = a \cdot x(t) \quad (a > 0), \quad x(0) = x_0 > 0 \Rightarrow x(t) = x_0 \cdot e^{a \cdot t} \rightarrow +\infty.$$

This growth is essentially stronger than linear growth. There is only the trivial and unstable steady state $x_s = 0$ ($x'(0) = a \cdot x_0 > 0$). The time of doubling x is in this case

not depending on t : $t_d = \frac{\ln 2}{a}$. This time can be very short or also very long. Hence

exponential growth spans a wide range of growth. This kind of growth is typical for reproduction of simple organisms. It also often occurs in certain growth periods of higher organisms. Malthus assumed in 1798 that also the human population on earth is growing exponentially. He predicted a catastrophe in future because not enough food would be available. He feared terrible wars for food. But against this expectation the growth slowed down without a catastrophe. Later other experts claimed a limit capacity of about 10^{10} for earth, which are 10 billion people. Considering information of US Census Bureau the human population grew faster than exponential until 1960 while later the growth slowed down. The numbers after 2000 are forecasts (see figure 2).

In [5, chapter 11] least squares fits are made in the class $S(t; a_1, a_2, a_3) = a_1 \cdot e^{a_2 \cdot t^2 + a_3 \cdot t}$ considering the time before 1960 and after it up to 2050 separately. In this case the growth rate is not constant, but time dependent. Generally it is

$$x'(t) = a(t) \cdot x(t), \quad a(t) > 0, \quad x(0) = x_0 > 0 \Rightarrow x(t) = x_0 \cdot e^{b(t)}, \quad b(t) = \int_0^t a(s) ds.$$

The formulas in this subsection hold also for negative growth rates. But then the process x is monotone decreasing and describes decline or shrinkage. Radioactive decay is a example for exponential decline ($a < 0$).

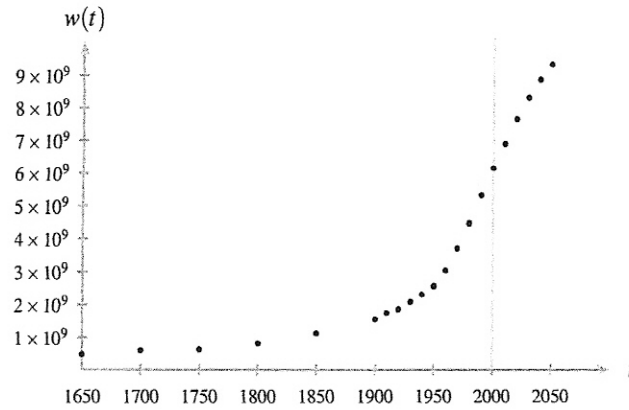


Figure 2. Development of human population (US Census Bureau, [5: p. 122]).

Bounded growth

Postulating a *limit capacity* C for growth, it is natural to assume that the velocity of growth is proportional to the difference $C-x$. If x is near the capacity C then the velocity of growth is small:

$$x'(t) = a \cdot (C - x(t)) \quad (a > 0), \quad x(0) = x_0 \Rightarrow x(t) = C \cdot (1 - e^{-at}) + x_0 \cdot e^{-at}.$$

Further it is $\lim_{t \rightarrow +\infty} x(t) = C$. Observe, that x is growing only for $x_0 < C$. By the way, x is of type 1 (see section 3).

Example: Suppose that in a region with C peoples information is spread by media. A simple model is that the number $x(t)$ of people which have the information at time t fulfils nearly the above relations. Evidently, contradicting the model, x attains natural numbers and is reaching C at finite time.

Further Examples: compensation of temperatures, diffusion in cells.

Logistic Model: Another idea is to combine the term x of exponential growth and the term $C-x$ of bounded growth multiplicatively:

$$x' = a \cdot x - b \cdot x^2 = b \cdot x \cdot (C - x) = a \cdot x \cdot \left(1 - \frac{x}{C}\right) \quad \left(a > 0, b > 0, C = \frac{a}{b}\right), \quad x(0) = x_0 > 0$$

$$\Rightarrow x(t) = \frac{C \cdot x_0}{x_0 + (C - x_0) \cdot e^{-at}} \rightarrow C, \quad r = \frac{x'}{x} = a - b \cdot x = \frac{a \cdot (C - x_0) \cdot e^{-at}}{x_0 + (C - x_0) \cdot e^{-at}} \rightarrow 0.$$

This leads to logistic growth for $x_0 < C$ characterized by three different periods. In the first period there is slow growth, in the second the growth strongly increases and in the third nearby the capacity the growth becomes again slow with tendency to stop. The growth curve is sigmoid with a turning point $t_w = \frac{1}{a} \cdot \ln \frac{C - x_0}{x_0}$ reaching

there halve capacity $x(t_w) = \frac{C}{2}$. Observe that t_w can be negative. After it the growth loses vitality. A standard curve is given in figure 1. If $x_0 > C$, the process declines monotone to C .

The logistic equation $f(x) = b \cdot x \cdot (C - x) = 0$ supplies the steady states $x_{s0} = 0$ and $x_{s1} = C$. Taking $f'(x) = b \cdot (C - 2 \cdot x)$ into consideration, x_{s0} is unstable because

of $f'(x_{s0}) = b \cdot C > 0$ and x_{s1} is stable because of $f'(x_{s1}) = -b \cdot C < 0$. Thus, starting with positive x the process $x(t)$ tends in time to the stable steady state C .

Example: The logistic approach is a simple model for spreading of information in a region with C people by passing it on orally starting for instance with $x_0 = 1$ person.

Further examples: reproduction of populations, growths of organisms and plants.

A lot of growth processes with limited capacities can be well approximated by the logistic approach. Two parameters occur in the solution class which can be fit in a least squares approach optimally to the given experimental data (see section 3). Also the growth of human population on earth can be approximated by a logistic function (see figure 2). Therefore some people believe that there is a general logistic principle. More likely it is the simplest way to model a lot of bounded growth processes.

Gompertz model: In a modified approach the term x^2 on the right side of logistic differential equation is substituted by the term $x \cdot \ln x$:

$$x' = a \cdot x - b \cdot x \cdot \ln x \quad \left(b > 0, C = e^{\frac{a}{b}} \right), \quad x(0) = x_0 > 0$$

$$\Rightarrow x(t) = C^{1-e^{-bt}} \cdot x_0 e^{-bt} \rightarrow C, \quad r = \frac{x'}{x} = a - b \cdot \ln x = (a - b \cdot \ln x_0) \cdot e^{-bt} \rightarrow 0.$$

Steady states are $x_{s0} = 0$ and $x_{s1} = C$. The first is unstable and the second is stable.

Growth arises for $x_0 < C$. A positive turning point $t_w = \frac{1}{b} \cdot \ln(a - b \cdot \ln x_0)$ is given if $x_0 < e^{\frac{1}{b}} \cdot C$ (type 2).

Growth Models With Several Steady States

The logistic model can be generalised by subtracting on the right side of the differential equation a rational function $p(x)$ reflecting a further concrete influence:

$$x' = a \cdot x \cdot \left(1 - \frac{x}{C} \right) - p(x), \quad x(0) = x_0 < C.$$

Since the differential equation is autonomous a solution formula can be given containing a , C , the coefficients of p and x_0 as parameters). The form of the solution can be strongly influenced by the parameter values. Small changes in the parameters can lead to qualitative changes of the solution. This will be illustrated by a special choice of the rational function, namely

$$p(x) = \frac{B \cdot x^2}{x^2 + A^2} = B \cdot \left(1 - \frac{A^2}{x^2 + A^2} \right) \rightarrow B \quad (x \rightarrow +\infty), \quad p(0) = 0.$$

This is a growth function of type 2 starting progressive and ending digressive with turning point $x_w = \frac{A \cdot B}{\sqrt{3}}$ (see section 3). The corresponding differential equation was used to model outbreak of spruce budworm population x defoliating fires in Canadian forests. The additionally function models predation by birds (see [4, section 1.2]). The predation is present after x has passed the threshold value x_w . A simplification of the

problem can be reached by substitution to get a dimensionless equation:

$$\tau = \frac{B}{A} \cdot t \geq 0, \quad u(\tau) = \frac{1}{A} \cdot x(t) \geq 0, \quad u_0 = \frac{1}{A} \cdot x_0, \quad r = \frac{A}{B} \cdot a > 0, \quad q = \frac{C}{A} > 0.$$

This leads to the simpler problem

$$u' = \frac{du}{d\tau} = r \cdot u \cdot \left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} = f(u; q, r), \quad u(0) = u_0 > 0.$$

The right side can be also written in the form

$$f(u; q, r) = \frac{u \cdot P_3(u)}{1+u^2}, \quad P_3(u) = u^3 - q \cdot u^2 + \left(1 + \frac{q}{r}\right) \cdot u - q.$$

The trivial steady state $u_{s0} = 0$ is not interesting and unstable. There can be one or three further positive steady states $u_{s1} < u_{s2} < u_{s3}$. The steady states u_{s1} and u_{s3} are stable while u_{s2} is unstable. Which stable steady state is reached for increasing time depends on the position of initial value. The situation is discussed if four steady states exist. The process u is monotone increasing to u_{s1} for $0 < u_0 < u_{s1}$ and monotone decreasing to u_{s1} for $u_{s1} < u_0 < u_{s2}$. If $u_{s2} < u_0 < u_{s3}$, then u is monotone increasing to u_{s3} . If $u_0 > u_{s3}$, then u is monotone decreasing to u_{s3} .

In [4] the dependence on parameters is discussed. The q - r plane can be divided into two regions with parameters leading to one or three positive steady states (see figure 3).

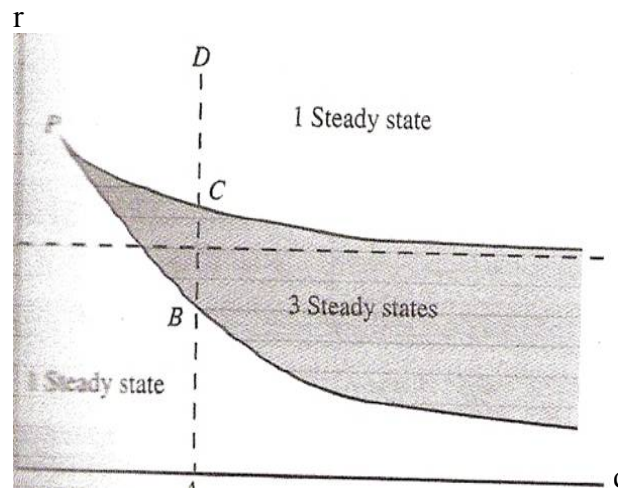


Figure 3. Parameter domain for the budworm model (see [4, p. 9]).

A *hysteresis effect* can be invented by fixing q and crossing the second region for increasing and decreasing r . The three positive steady states characterize in natural order the *refuge*, the *transit* and the *outbreak equilibrium*. A reasonable strategy should be to minimize damage of fir by the spruce budworm that is to prevent the outbreak equilibrium by starting with small x_0 and by reducing r or q , respectively. An important question is to find out which practical steps lead to such a parameter change. Finally, the model shows that already small changes of parameter values can cause big changes in population development.

Exercises Concerning Growth Processes

The training of problem solving is very important in engineering education. At the beginning simple practically or theoretically oriented exercises are useful. Three examples are given below:

Exercise 1: A colony of bacteria is given into a culture liquid at time $t=0$. Suppose that the colony consists of 329 bacteria after 30 minutes and of 2684 minutes after 60 minutes. Using the exponential growth model determine the time of doubling the population. How many bacteria are in the colony after 5 hours?

Exercise 2: R. Pearl proved by experiments in 1920 that the growth rate x' of a fly population with x members (*Drosophila*) satisfies approximately the equation

$$x'(t) = \frac{1}{5} \cdot x(t) - \frac{1}{5175} \cdot x^2(t),$$

where t is measured in days. Assuming $x_0 = 10$ flies at the beginning determine the population function $x(t)$ and the carrying capacity C . How many flies exist after 12 days? At which time the growth rate is starting to decrease?

Exercise 3 [4: p. 40]: Starting with the dimensionless budworm model (see section 5) containing parameters p and q show that the separating curve in r - q space of figure 3 is given parametrically by

$$r = r(a) = \frac{2a^3}{(1+a^2)^2}, \quad q = q(a) = \frac{2a^3}{a^2-1}.$$

Show that the two parts of the curve meet at a cusp for $a = \sqrt{3}$. Sketch the curves in r - q space noting the limit behaviour of $r(a)$ and $q(a)$ as $a \rightarrow \infty$ and $a \rightarrow 1$.

Activities and Reactions of Students

The topic of growth offers many possibilities of project work in higher education where general questions in society and science can be connected with theoretical background using mathematical basics. A selection of topics is given:

- Political and philosophical questions connected with growth processes
- Study of function families representing different qualities of growth
- Collection of statistical data concerning real growth processes; optimal data fit in appropriate function families
- Modelling of growth processes by differential equations of first order, determining the meaning of parameters in the real context
- Investigating growth processes by application of calculus (derivatives, local extremes, turning points, monotony intervals, convex and concave curve segments)
- Calculating steady states of growth processes by using solution methods of nonlinear equations
- Studying elementary solution methods of differential equations of first order (analytical and numerical methods)
- Solution of growth models using different parameters and comparing results
- Solution of growth models using different solution methods and comparing results

- Collecting and solving theoretical and practical exercises based on mathematical models of growth processes
- Use of MATLAB to produce problem solutions applying graphic, numeric and symbolic means; comparing results with results obtained by manual calculation or given in the literature.

There are a lot of textbooks containing appropriate material concerning mathematical tools and application oriented exercises (see section 6 and e.g. [2, chapter 2], [8, chapter 7]).

Up to now the project of growth processes was only tested in an optional course. The reactions of students were very positive in general. Asking for there estimation the following answers were obtained (translation from German language):

- The project is innovative and increases my motivation to learn mathematics.
- The project involves interdisciplinary thinking, problem solving, modelling and simulation. Hence the practical solution of problems is trained.
- The project stimulates discussion about general problems of society. So it contributes to the awareness of own responsibility.
- The project is more challenging and activating than classical lecturing.
- The project supports independent and responsible work as well as cooperation.
- I can use computer software and resources from the internet. This extends my knowledge considerably.
- I like team working with my friends.
- The project was interesting. But I would prefer projects which are concerned with problems of electrical engineering.

The majority of students had profit from the project (> 90%, motivation, knowledge, performance). Also staff had some benefit. A lot of additional material was collected by the students.

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Perceived Route to Success by Pasifika Tertiary Students

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Our work with university Pasifika mathematics students has revealed that frequently these students do not follow a clear plan that will lead to the completion of a degree. This paper examines if students have a preconceived path through university study. We investigate to whom they turn to for advice and guidance when making their university career choices, if they have a plan of study and if they understand how to negotiate the requirements for completion. The information obtained from this research will help tertiary mathematics departments devise strategies to effectively steer students towards recognising their goals and how to choose courses that will maximise their chances of attaining them.

Keywords: Pasifika; guidance, decision-making, mathematics

Background

In this paper we investigate if Pasifika have a preconceived path through university study and examine if they are aware of the consequences of their choices. Do Pasifika choose courses that relate to specific career goals, and if not, what other factors do they base these choices on? We would like to know whether or not students plan ahead – in terms of their academic programme, and in terms of their career aspirations and who advises them.

Statistics New Zealand (2006a), state that 6.9% of the New Zealand population were Pasifika. Pasifika New Zealanders are a young population with median age of 21 years, and a break down of the population of 2006 data gives:

7.8% of 20-24year olds were Pasifika

8.4% of 19 year olds were Pasifika

Through the statistics available on the websites of the following universities we estimate the proportion of Pasifika students in 2010 to be:

7.8% The University of Auckland

5.2% Victoria University of Wellington

3.3% University of Waikato

3.1% University of Otago

2.2% University of Canterbury

This indicates that less than 6% of New Zealand university students are Pasifika.

Of even greater concern are the graduation rates. Table 1 is based on data on the Statistics New Zealand website (Statistics New Zealand, 2006b; 2006c), and gives the percentage of all New Zealanders of 15 year olds and over with degrees in 2006, compared with the percentage of all Pasifika of 15 year olds and over with degrees.

Table 1. Proportion of Population With Degrees

	Bachelor or level 7 equivalent	Masters	Doctorate
New Zealand	11%	2%	0.5%
Pasifika	3.4%	0.4%	0.05%

The low proportion of Pasifika with degrees must lead to under-representation in the professions that require university qualifications. It is important that Pasifika are well represented throughout all the various professions. This representation is important for providing role models for younger Pasifika students and for making the best use of the country's full range of potential. Our study examines the barriers that affect the progress of Pasifika through to the completion of university degrees.

The Department of Mathematics at our university had developed a programme for its Māori and Pasifika students that comprises a number of initiatives that were introduced over the past 10 years. This programme was developed to encourage enrolments, retention, participation and motivation at every level for Pasifika. A cultural study space was made available where students could access computers and other resources. In 2001 a mentor programme for stage 1 students was introduced where senior students were assigned to small groups of Māori and Pasifika students to provide mentoring, tutoring and to provide a sense of cultural belonging within the department. The programme was developed over time and other smaller-scale initiatives have also been introduced, including 1-1 tutoring of stage 2 and 3 students.

Since the introduction of this programme pass rates were monitored in order to assess the impact of these initiatives. The success of this programme measured in terms of student pass rates is unclear, due to a number of outside factors that also had an impact (and possibly a larger impact) on the programme and on pass rates. For example, changing personnel and their competing commitments. But overall it is believed the programme has had a positive impact.

Anae, Anderson, Benseman and Coxon (2002) identified some of the socio-cultural and socio-economic barriers to success for Pasifika students. These barriers were specific instances of personal hardship such as family pressures, health and housing issues, lack of personal motivation combined with the relatively easy-going lifestyles and also institutional barriers of unfamiliarity with the university environment and academic work. Such barriers were discussed and recommendations were made for support systems and clear pathways within schools and tertiary institutions. Ethno-mathematics educators have examined the language of mathematics itself as a possible barrier to minority students learning in mathematics while others have examined the nature of mathematics itself as a possible barrier to minorities succeeding in mathematics (Barton, 1995; and Latu, 2005).

Anecdotal evidence gathered from our regular interactions with mathematics students indicate that many Pasifika do not follow direct paths to the completion of a degree, in stark contrast to other students. There are a large number of Pasifika who appear to change direction either after failing courses or for no obvious reason. For the majority of Pasifika in our study their main motivation is to get a degree. While this is not surprising, the degree is the end point, the goal, and from the students we speak with, there seems to be little thought about what career they are working towards following the completion of their mathematics degree.

Communication seems to be a recurring issue for Pasifika - both between the students and lecturer as well as between the university in general and the students. For tertiary educators there is a responsibility to ensure the success of their students. It is important therefore that educators assist their students in making decisions that will lead to their success. From previous interaction with new undergraduate students we realise they expect close relationships with the university and their lecturer in particular, as they had in school. However, communication needs to work both ways i.e. students must be prepared to ask for help if it is required. Several students consider asking for help as being quite intimidating as at times, students were unsure

of what question to ask never mind whom they should ask. The situation is amplified within the field of mathematics where it can be difficult for students to ask questions when a lack of understanding of a concept can mean that they do not have the language to formulate a question.

Literature Review

There have been many reports and articles written about Pasifika students at university; about the extent of their participation and what influences and motivates them. For example, McAllister (2008) wrote a comprehensive review of literature that investigates the influencers, motivators and barriers for Māori and Pasifika to study at university level. Anae, Anderson, Benseman and Coxon (2002) compiled a report on participation of Pasifika students in universities. Hattie (2003) also considered factors that could influence Pasifika in their pursuit of educational success. They were in accord that the main factors influencing motivation were self-image, finance, home and community, school environment, preparations for tertiary and tertiary experiences.

This paper considers a more specific area that can influence motivation – i.e. help and guidance for Pasifika students when making decisions about their own tertiary education. It seems there is a limited amount of research in this area. Leach and Zepke (2005) wrote a literature review on students of all ethnicities and their decision-making – how they make decisions, what factors influence them, the acquisition and sharing of information and how their decisions are affected by diversity.

Hattie (2003) believes that it is the teacher not the student or the environment that will have the most significant effect on the achievement of the student. Anae et al (2002) strongly agree with this and write that Pasifika students prefer Pasifika role models who can offer advice about all aspects of tertiary education and where to go for help in specific areas.

Not all students are aware of the different pathways to university, the differences between similar courses or the implications of attending one institution over another. The students are reliant upon others to help advise them about which course or which institution to choose and poor or inadequate guidance can lead to students withdrawing from courses or not achieving.

When it comes to finding information about tertiary studies and careers, Pasifika at secondary school often feel that there is not enough information available to them (Anae et al, 2002; Benseman, Coxon, Anderson and Anae, 2006).

Very few participants reported that they enlisted the help of their guidance counsellors or career advisors from their schools. Those who did seek assistance did not find them at all helpful. (Benseman et al (2006), p.80)

Anae et al (2002) found that the majority of secondary students did not meet with the guidance counsellor, for a variety of reasons. The quality of careers guidance is another factor that may influence students. Pasifika students often struggle to find relevant information to help them make informed decisions about their academic choices. The situation can only be exacerbated if they do not approach guidance counsellors or if they are given poor advice.

When Pasifika students are making decisions about what subjects or where they wish to study, one of the more influential factors is people - mainly family members (Schagen and Hodgen, 2009). Although they are at ease talking to these people, they may not receive the best advice or correct information. Leach and Zepke (2005) reiterate this and say that information obtained through interpersonal relationships is

found to be more effective than information aimed at groups.

Students may also face excessive pressures from family to study particular subjects and to strive for particular careers, often in areas or to levels that are not suitable for the individual. This is often due to the earning capacity of the particular career or the kudos that would accompany it and be shared with the family (Mara, Foliaki and Coxon 1994). Leach and Zupke (2005) found that decisions by minority ethnic groups to pursue university education are made early and are encouraged if family members have personal experience of tertiary education.

The outcome of the decision depends on whether the family has the social and cultural capital to envision an educational path for their children and, more importantly, whether they aspire to obtain the social and cultural capital offered by tertiary education. (p.27)

They add that social and cultural networks are essential formats for families and supporters to gain access to useful information about tertiary education. This is in accord with Mara et al (1994) who recommend the introduction of an information service for Pasifika run by Pasifika. If this service was available throughout the Pasifika community better choices surrounding careers and tertiary education could be made. This is also reiterated by McKegg (2005) who writes that all students need to feel connected to their cultural group. If there are role models with whom they can compare their experiences and if they can meet the various personnel they will encounter during their time at the institution, this will increase levels of motivation and help them make informed decisions. Mara et al, (1994) says the presence of role models and an awareness of cultural identity is essential for the students to know what to aim for and know where their aspirations lie within their own culture and society.

Anae, et al (2002) report that many students adopt significant self blame attitudes and although there are many inspiring role models of Pasifika people who have succeeded in tertiary study, their prominence is limited. It is a particular challenge to bring the positive image to the fore to counter personal issues of motivation, persistence and self-belief. Mara et al (1994) says that the thirst for knowledge is present in Polynesian parents and students and the presence of role models and an awareness of cultural identity is essential for the students to know what to aim for and know where their aspirations lie within their own culture and society. Strengths must be built on but the context must be provided where students can achieve academically as well. Mara et al (ibid) believes it is essential to involve parents and the Pasifika cultures in academic life.

The literature indicates that Pasifika students place great emphasis on the value of Pasifika providers. With the low participation rates and low pass rates within the Department of Mathematics the Pasifika students do not have access to Pasifika role models, which creates an extra challenge.

Methodology

For our study we selected students from the database from a New Zealand university who identify themselves as Pasifika and who enrolled in a mathematics course at the university during the first semester 2011. In total there were 146 students on the database that matched these criteria. The data collection consisted of a questionnaire with a follow-up interview for those students who had completed the questionnaire.

Questionnaires containing 16 questions were sent out to each of these 146 students. There was a mixture of multiple choice questions as well as open questions

and an opportunity to provide any other comments or observations they wished to make.

The students who returned the questionnaires were contacted to arrange an interview. In order for the students to feel comfortable and at ease, a postgraduate Pasifika student who was not connected with the Department of Mathematics conducted the interviews. She was able to interview six students.

The questionnaire responses and interview transcripts were analysed qualitatively by each researcher to identify emergent themes. Through discussion these themes were formalised and codes developed. The data was then reviewed in greater detail with these codes and themes in mind and we were thus able to determine actual trends within the data.

Results and Analysis

Our study aims to understand whether or not Pasifika students taking mathematics plan ahead - in terms of their academic programme, and in terms of their career aspirations. We intend to establish to whom they turn for advice and guidance when making their university career choices, if they have a plan of study and if they understand how to negotiate the requirements for completion.

From our data we identified five emerging themes: expectations of attainment, forward planning, advice from a friend, taking responsibility, and decision crossroads. We relate our findings using these five categories.

Expectations of attainment

The majority of students in our study spoke of instances where their initial expectations had to be revised because they had been unrealistic in what they anticipated. They had hoped to follow a specific academic pathway but their grades at school or university hindered their plans.

Students A: I actually applied for [another university] and I considered doing mathematical science there but initially I wanted to do engineering but I didn't get in so I went for science. I was going to try the alternative pathway but then I changed my mind... [I'm aiming to study] a Bachelor of Engineering in IT. I'm actually doing Science at the moment. If I can get in after this semester then I'll change, if not then I'll try again after second semester.

Forward planning

Students in our study frequently spoke of instances where they had no clear path or goal. They seemed to consider that university was a goal in itself rather than a stepping-stone on the longer journey to a career.

Student A: I kind of knew that [my degree] would be science, something science related. I didn't think of what kind of major because there was so much. I knew it would be maths or I.T something like that and so I just left out all the commerce and law and arts and all that.

They also spoke of making wrong choices for their university courses and plans for their degree programme. Reasons for making these wrong choices seem to be based on incorrect advice from a variety of sources, impulsive decisions and lack of direction and planning.

Student B: I think it was more of an impulsive thing because I'm currently in that field, but my heart wasn't really there.

Student D: I just took it year by year. I didn't map out a 3-year thing. But then I got through some of them and thought damn if I knew I could have done that one earlier, I would have. ... If I had planned better, including summer school, I could have finished before 3 years.

This is confirmed by Anae et al (2002) and Benseman et al (2006) when they describe the effects of Pasifika not seeking assistance or not being able to find appropriate advice.

Advice from friends and family

One of the questions students were asked was where they have sought advice from in order to help them make decisions. Many of them describe talking to friends as the most helpful source of information, although they also discussed the different courses they were considering with family members or friends who had already taken the course.

Interviewer: So who did you used to seek some help or advice from in the past?

Student A: Probably students, my careers advisor was really good. Yeah, students who were already at uni. ...if they've done the paper it kind of gives you a fair idea of how hard or easy it is.

One student spoke of the conflicting advice she was given by different family members. She was obviously trying to please everyone by taking a mixture of courses.

Student C: My mum said to carry on with my studies, ... my aunty said do science and that's what my dad said but I want to do commerce, that's what my mum wants.

Interviewer: What do you want to do?

Student C: I want to do commerce but I want to do science as well because that's what my family wants.

Another student spoke of relying on the advice from friends who suggested attending a lecture the previous semester to try out the course and lecturer.

Student D: I ask [my friends] what they're doing and how they liked papers.... One of my friends was like, go into one of the [lectures] for your papers in next semester and see if you like it. I mean its just one lecture but it'll give you an idea about it, if its something you hate.

This is in agreement with Schagen and Hodgen (2009) and Leach and Zepke (2005) who write that students are influenced by family members and close friends.

Taking responsibility

The students in our study spoke about taking responsibility for their situation, and in each case how their decisions were influenced ultimately by someone or something else.

Student B: ... I didn't do my tertiary education immediately after high school, if I had been given the right directions, ... I think it was because of my circumstances at the time, which was the injury.

They seem to be attempting to take the responsibility for their decision-making but there always seems to be a factor beyond their control that dictates their level of success.

Student B: I find that I'm not fully focussing, because the financial pressures are always there.

Student C: ...because I always go late to class because I have geography 101 and then I have to go all the way to the [other] building for the lecture, so I get there late and I miss a lot of notes...

Student D: ... I'm probably behind a month on all of them but workload's hard because they make the due dates for the papers I've got all in the same week, so instead of going to classes I miss a week of classes trying to complete assignments.

As Anae et al (2002) write many students adopt a self-blame attitude and although they may not be entirely blameless, they are lacking role models and confidence in their own abilities. This may create an environment where it is not easy to succeed. Mara et al (1994) agrees that role models are vital for the students to match up their cultural identity with their future career aspirations.

There is no reason to suggest that this phenomenon does not apply to any student who does not fulfil their personal expectations, but we include it here as it has the potential to create further barriers for their progression through university.

Decision crossroads

One student received conflicting advice from different members of her family. This left her unsure of what to do next.

Interviewer: So you're doing science, do you want to go back to commerce?

Student C: Yes, or I might do a double major

The same student received the wrong advice at enrolment but did not query or question the error.

Student C: ... I asked the girl to enrol me to stats but she enrolled me to maths cause she thinks, I think, she thought math was the same as stats.

Despite the error at enrolment she persevered with the course even though it was not the area she had experience in. She did not enjoy the course and it left her in a state of confusion, and not knowing what to do next.

Student C: Oh, I don't know because I want to change my degree again. I want to change my major but I'm not sure.

Facing a decision crossroads these students need reliable advice from people whom they have built a relationship with and who also have the correct information available (Anae et al, 2002; Benseman, et al, 2006). The presence of role models is essential for Pasifika if they are to achieve success at university.

Discussion

It is apparent from our study that some Pasifika students studying mathematics at university struggle with decision-making regarding their academic programme and future career. They are reluctant to ask for help but even when they seek help those advising do not necessarily take the time to understand the student's situation, or the advice is unclear as it presumes a level of understanding that students do not have.

Several students in our study did not have specific goals. It was as though university was the goal but it has not been thought through. Some have an idea that they want to do engineering or commerce, but there is also a lack of clarity about what that means.

Their choices on arrival at university are frequently based on shaky ground: recommendations from friends about specific papers; incorrect enrolments that are left unchecked; taking science because they do not get into their first choice, with the belief that they will be accepted in another year; and believing they want to do a particular subject, but then changing their mind the following year or semester. There is evidence of a lack of understanding of completion requirements – none of the students interviewed were able to state how many credits they require, and there are several examples of unfortunate consequences to their past choices.

Some students are entering university with an expectation that they will eventually complete a degree in a field that they do not on entry have a sufficient level of attainment or preparation to be accepted into, or an expectation that they will reach a level that, based on their background, would require an enormous investment in time and study, such as a double degree. A concern is that, rather than direct their efforts toward a degree that they have the preparation for, they may be in danger of taking a path with an outcome that is too uncertain.

Initially, students appear to identify personal factors that influence their situation but further investigation indicates that they are able to identify external factors as well. In some cases students believe the external factors were more responsible for their situation. While external factors may have a strong influence on their decision-making and outcomes, we believe that personal factors are just as important and that taking personal responsibility is essential for success.

The key seems to be one-on-one course advice to students right at the start. The advice needs to be personal with time taken to properly understand the students' background and aspirations.

The lack of role models for Pasifika students is also a significant issue. We expect that the low proportion of Pasifika with doctorates will have an impact on the proportion of Pasifika academic staff throughout New Zealand universities. The need to reach out and make contact with Pasifika on a more personal level is paramount. However the likely paucity of Pasifika staff poses a major challenge to achieving this objective.

It is important that Pasifika achieve success at university. If they are not able to fulfil their potential then New Zealand is also losing that potential. By increasing representation in the professions, there will be more role models and the Pasifika input into these professions will have an impact on future generations in achieving equity.

Our study is based on a small number of questionnaires and interviews with Pasifika undergraduate mathematics students and we do realise that this cannot be taken as a representative sample from New Zealand. We focus on mathematics students but we believe our findings may well hold for all students. Mathematics poses its own set of challenges and we believe the statistics quoted become further exaggerated when considering science and other fields requiring mathematics. We believe this study provides evidence that there are issues surrounding decision-making by Pasifika students that require further investigation.

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A Decision Making Model Of Contingent Teaching Enabled Through Classroom Response Systems

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Classroom Response Systems (CRS) are now commonly used in many teaching institutes around the world. Over the past decade many researchers and educators have discussed and shown their effectiveness and potential in changing the way we teach and interact with students. In this paper we construct a teaching model by combining theories based on Scheonfeld's [1,2] highly interactive decision making routine and Resources, Orientations and Goals (ROGs), Draper and Brown's [3] contingent teaching and Beatty et al.'s [4] Question Driven Instruction (QDI). The model illustrates a teaching routine which is flexible; contingent to students' needs; makes use of the most updated information from students' feedback via CRS and benefits from teacher's decision making at appropriate moments. In the light of the proposed model we examine a case study using CRS in a Bayesian statistics course and suggest ways in which the teaching can be improved.

Keywords: classroom response systems; clickers; contingent teaching, decision making; audience response systems.

Introduction

In recent years there have been discussions that sitting and note-taking in passive mode in large lectures are not effective [5]. As Jensen [6, p. 60] declares "it's truly astonishing that the dominant model for formal learning is still 'sit and git' ". Certainly there are lecturers who try to get students engaged and involved in lectures by asking questions, but realistically this is not as feasible in large lecture halls. Typically there are some actively engaged students who are often better achievers, while the majority of the class are passive and without a voice and often fail to realise their potential. It would be difficult to know where students are at their learning, no one can be sure of this. However, it is possible to have a sense of students' thinking by means of using classroom response systems to allow instructors to rapidly collect and analyse responses during class [7,8]. These devices, known as clickers, are often used to create an environment that helps to support, deepen and enhance learning through promoting interactivity in large lectures [9]. Although the technologies date back at least to the 1960's [7], based on a comprehensive review of the literature by Kay and LeSage [10, p. 820] "the extensive use of CRS began in 2003". A question then asked by Abrahamson [11, p. 1] was "why, for an idea apparently more than forty years old, it took this long to happen!". His research shows two contributing factors for the late acceptance of the technology: the inadequacy of the technology itself and no gain in student achievement in the late 60's and early 70's.

However, the recent review by Kay and LeSage [10, p. 822] categorises many benefits of the use of clickers as follows: "attendance; anonymity; participation; engagement; learning benefits; interaction; discussion; contingent teaching; learning performances; quality of learning; assessment benefits; feedback and formative". Bruff [7] in his book "Teaching with classroom response systems" focuses on

teaching rather than the *changing* technological devices and gives a practical guide for instructors by incorporating case studies of interviews with almost 50 instructors from different disciplines. Draper and Brown [3] also support the idea that pedagogy should be put first and technology second. Thomas and Chinnappan [12] in reviewing 135 papers on use of technology (dated between 2004-2007) in teaching and learning mathematics found that “there is a need to develop effective methodologies for the integration of how technology is and can be used during instruction” [12, p. 184]. They also found that many methodologies lacked innovation. In Banks’ [13, p. ix] view “the technology, in itself, does not offer some wonderful new ‘magic bullet’ that will offer learning gains simply by its adoption. It can certainly provide novelty and fun for all participants, but must be used within the context of the teaching and learning process for its full promise to be achieved”. Draper and Brown [3, p. 93] believe that “success is associated with increasing the interactivity of the occasion”.

Theoretical Framework

A thought provoking way of using classroom response systems is through Question Driven Instruction (QDI). This perspective relates to methodology not the content. It is “...a cyclic process of question posing, deliberation, commitment to an answer and discussion” [4, p. 100]. The QDI cycle ends with a discussion and finally a group response via clickers. Research shows that although the instructors appreciate the feedback that they receive from the students [11] the challenge would be whether the lecturers can react to these responses on the spot, according to students’ needs. This is ‘contingent teaching’ [3] and requires the interactivity of lecturers as well as students. In this setting “a lecturer must develop their plans beyond the factory machine stage of executing a rigid, pre-planned sequence regardless of circumstances” [3, p. 91] and have more strategies on hand, based on students’ responses. In their belief no class of students are ever the same. Although the term contingent teaching is mentioned and described in the literature [3,7,14 ,15,16], there is no research on *how* this is done and detailed methodologies seem to be very rare.

One crucial aspect of contingent teaching would be that the instructor is constantly confronted with decision making. As Schoenfeld [2, p.36] points out “the quality of people’s decision making in problem solving, teaching, and most everything we do affects how successfully people attain the goals they set for themselves”. Although decision making is complex, nevertheless this comes naturally from one’s Resources, Orientations and Goals [2]. By resources he focuses mainly on knowledge, which he defines “as the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks” [2, p. 25]. Goals are defined simply as what the individual wants to achieve. The term orientations refer to a group of terms such as “dispositions, beliefs, values, tastes, and preferences” [2, p. 29].

While CRS is already useful, to increase its effectiveness a teaching model was constructed. This model originates from Schoenfeld’s [1] interactive teaching routine and incorporates CRS with Beatty et al.’s [4] QDI theory and makes complete use of Draper and Browns’ [3] contingent teaching ideas. Figure 1 illustrates a schematic view of the model.

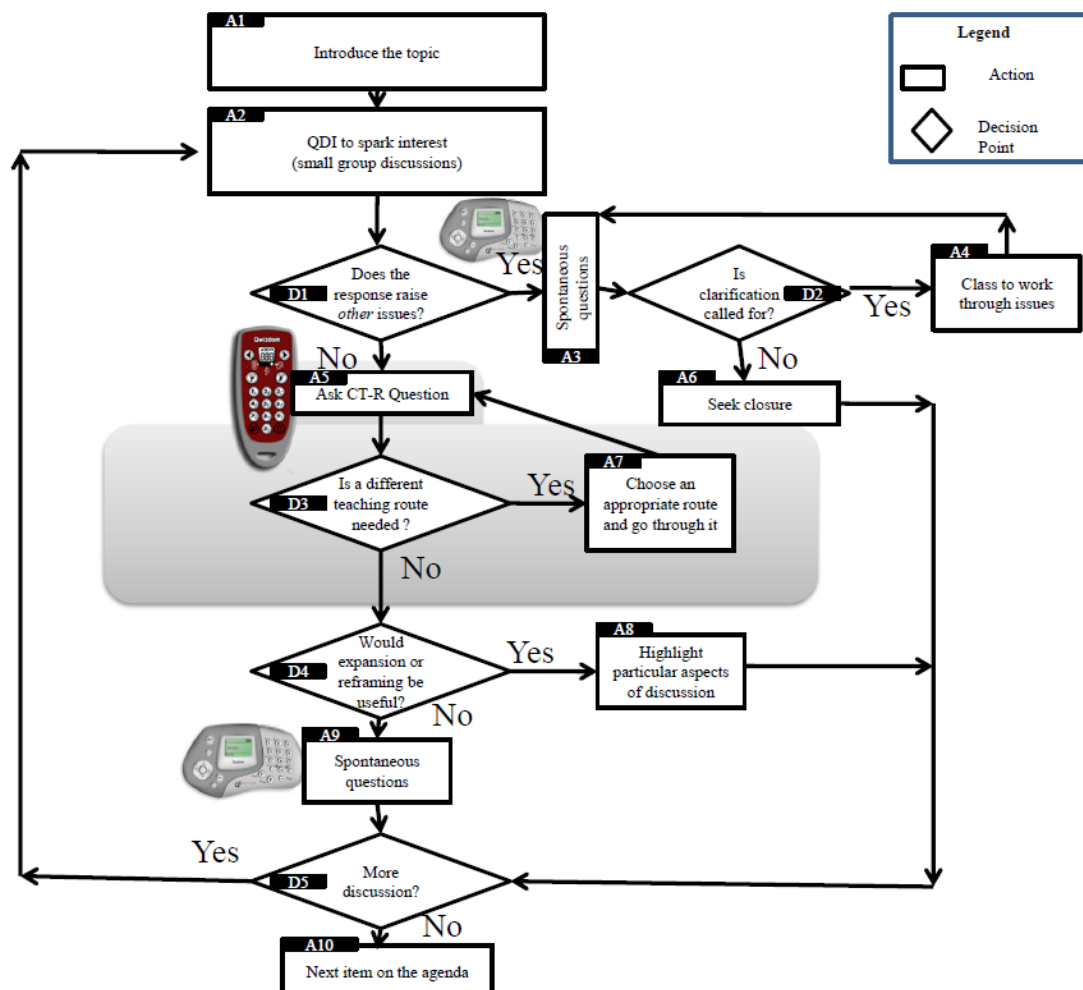


Figure1. The model- a modified version of Schoenfeld [1, p. 138] teaching routine for discussing a topic or part thereof with CRS.

How the model works

The above routine is a variation of Schoenfeld's [1] teaching model and uses the same notation and schematic. The rectangles are labelled A1-A10 and represent possible actions performed by the teacher. The diamonds labelled D1-D5 represent points within the teaching sequence where a decision must be made. The routine starts with an introduction of a topic or part of it [A1]. In order to help guide the students in their learning, Question Driven Instruction (QDI) begins in A2 and is done in the context of group discussion, case studies, tactile investigations or experiments. During this period the teacher will listen to conversations among the groups. As part of the QDI the teacher will need to come to some appreciation of *other* issues. They will in general be issues that are not predictable and can have a surprise aspect to them. The teacher will need to decide whether there are issues to be dealt with [D1] and if there are, find the distribution of students with these issues using spontaneous clicker questions [A3]. This can be done by writing down a list of issues on the board and asking students which category they fall into. The class will engage [A4] until closure [A6] has been decided by the teacher [D2]. Once closure has been achieved the teacher will decide in D5 whether more discussion should proceed. This will normally be "Yes" unless time or curriculum objectives do not permit it. The question [D1] will this time likely lead to a "No" decision concerning *other* issues and the CTR

(contingent teaching routes) question can be posed in A5. This begins a portion which is at the heart of the teaching sequence and will require careful well thought strategic questions. The slide used in A5 will contain a CTR question as well as hyperlinks to a number of slide sequences (routes). The answers given by students will be retrieved dynamically and thus instant feedback will be achieved. A decision then has to be made [D3] as to whether a new path of teaching should be taken or not. Although the routes will be contingent upon students' responses to the questions, each route [A7] will be created with the goal to move students to the main aim of the concept taught. This may result in many hyperlinked PowerPoint sequences (routes) which will not necessarily be used. After the grey zone has been negotiated and significant difficulties addressed, D4 is invoked by the teacher. The issues that have not been discussed in detail may be covered less formally in A8.

Clearly, this model with some minor changes in the routine (e.g. more action boxes with different sequences), rests on Schoenfeld's [1] interactive model of teaching. However, it makes use of CRS to take consideration of the *whole* class to make more accurate decisions and uses prepared lesson material in the form of different routes (extra teaching sequences of content) contingent on feedback received from the students.

Our research question is: How does the model help us to analyse and improve teaching?

Method

In January 2009 for the first time, the University of Auckland implemented a trial run of CRS in lecture rooms over the summer semester. The authors together with the head of mathematics department at the time were among three initial pioneers who started lecturing (introductory data analysis and mathematics courses respectively) using clickers. Despite many challenges such as: remembering certain routines; setting correct options; memory overload caused by many programs running simultaneously, and other difficulties the trial was technologically successful. After the trial students were given the opportunity to borrow clickers through the university library system in semester one 2009.

The study presented here is a case study and was carried out at the University of Auckland in the second semester 2010. The second-named author (referred to as 'the lecturer') was teaching a third year undergraduate Bayesian course, which is a competing paradigm with classical statistics, using CRS. They were used in the form of quizzes (one per lecture, each including 3-4 questions) in every lesson to check assigned readings and understanding of the concepts (current and previous days). There were 100 students (mainly statistics majors) enrolled in this course. The use of these devices was compulsory and students received 2% of their final grade for answering clicker quizzes during the lectures. The lecturer had a participant list and students' responses together with their grades were recorded. The grade for quizzes was made from their best 20.

Although students were involved in group discussions, the teaching was not based on the model since its development came later. At the end of the course a small number (six) of students volunteered to be interviewed (we refer to them by the following names: Reuben, Josh, Jordan, Emily, Sara and Cathy). Also 17 students volunteered and filled out an attitude questionnaire as a way of giving some feed back on the use of clickers (see Table 1 for the list of questions). The interviews were semi structured and audio taped and later transcribed. The questionnaire responses were put in a bar plot and analysed. Moreover, the lecturer wrote a diary of day to day events

and kept a record of all the clicker questions and students' responses (bar plots produced automatically by the software). The purpose of conducting this case study was solely to examine the effectiveness of CRS in teaching and students' attitude toward using them.

Table 1. Interview Questions and the Questionnaire

Interview questions	Questionnaire
Did the use of clickers help you to understand the course material? Explain.	1. I believe more lecturers should use clickers.
What were the strengths of using clickers for you?	2. I like the instant feedback that I receive from clickers.
What were the disadvantages of using clickers?	3. Overall, I like using the clickers.
How could clicker use be improved?	4. Having the answers discussed helped my knowledge of the subject.
Was it a waste of time? If so, in what way?	5. I like to see how well I do relative to the rest of the class.
Were multiple choice questions a good way to check your knowledge or could a better style of question be used?	6. The fact that no-one knew who was voting except the lecturer was important to me.
Would you like the term test to be carried out using clickers? Why?	7. The clickers kept me awake and alert.
This year clickers were used as an assessment tool, did you find this fair?	8. The clickers used up too much time which could have been used for more important things.
	9. Clickers were hard to use.
	10. Overall, I found the clickers helpful in the task of learning the course.
	(a modified version of a questionnaire by Hinde and Hunt [15, p. 144])

Results and Discussion

In this section we will present a sample of clicker quizzes, students' responses and attitudes and discuss the issues around them. We will also suggest how the teaching could be improved based on the model.

Clicker quizzes

In the following statement the lecturer clearly maps out his resources, orientations and goals for the course and describes why there is a need for daily quizzes which will count towards the final grade:

In order to achieve a transition to Bayesian thinking students must first understand the chief characteristics of the classical paradigm. Therefore a simple problem in classical statistics was carried out and deconstructed into the main classical theoretical parts and assumptions, including a thorough discussion of the conceptual underpinnings. Students' understanding was assessed using clickers daily and problems discussed and treated as the course developed. Once they were reminded of the classical methodology the Bayesian paradigm was introduced and compared. The hierarchical structure of the curriculum demanded that all important items were understood by the student before moving on. Hence, continual assessment via clickers helped alleviate problems. A major goal of the lecturer was to guide students

to a better qualitative understanding of the differences between the paradigms and understand their philosophical importance.

In his diary the lecturer wrote:

Today I used 5 review clicker questions (True/False) – they were very effective in discovering the students’ weaknesses in regard to Triplots and Bayes’ factors.

Here is an example of one of the questions:

Quiz A. The plot (see Figure 2) is asking the students to interpret the prior and posterior at $\theta=3$. The heights are identical and therefore the data (through the likelihood) has not made us believe $\theta=3$ is any more likely.



Figure 2. Quiz A and students’ responses.

Clearly, the ideas displayed in the triplot were not as obvious to students as 45/66 (68.2%) of the students were correct while 21/66 (31.8%) gave an incorrect response. These were highly surprising results to the lecturer given his prior belief (orientation) that the plots and underlying ideas were straightforward. What the results showed was that almost a third of students had difficulty understanding the Bayes’ box, the heart of what they are expected to know in a Bayesian course.

Thus, the lecturer reviewed the material and tried his best to fix the problem on the spot. It seems that in lecturer’s opinion the quizzes were a good way of revealing students’ misconceptions.

However, what he could have done is to have other material prepared in advance (a different teaching sequence of slides) and acted accordingly contingent to students’ responses.

Nevertheless, doing quizzes and the instant feedback seemed to help students:

Cathy: ...because of the clickers I memorised the Bayes box before the term test and it was because of that part I passed the test otherwise I would have failed. Because the Bayes’ box is really important for this paper. Most of the basic knowledge is from it.

Josh: I think it's more we get the results more instantaneously. I think that's the strongest point.

Emily: Yeah it made me actually go and revise my notes when I knew a quiz was coming. So yeah it helped deepen the understanding of what he had been going through in previous lectures.

Here is another question that the lecturer posed:

Quiz B. The question (see Figure 3) is asking whether the students understand ideas of Bayesian prediction and WinBUGS coding. Munew is a mean logical node whereas ynew is a normal stochastic node with munew as its mean. It has therefore two sources of variability, munew and the normal compounded, ynew will therefore have the greater variability.



Figure 3. Quiz B and students' responses.

This question was more complex than the previous one. It was poorly understood (31/58 (53%) gave incorrect responses) despite repeated explanation on the previous day. Therefore the lecturer immediately spent considerable time and effort to re-explain the material with a different perspective and analogy. More than half of those who responded, did not give a correct answer. From the teacher's point of view it is marvellous to have this feedback immediately. However, it would be ideal to predict these situations and be prepared before hand and give a more comprehensive explanation.

Commencing with quizzes which were contributing 2% towards the final grade meant that the attendance increased and was maintained.

Reuben: I would have an incentive to attend class more. Because there was about 2% weighted on your attendance through the clickers - or getting the questions right... Maybe he could bump it up to 5% even.

However, after a couple of weeks the lecturer realised that starting the class with quizzes is disadvantaging the late comers.

Lecturer: "One problem I have noticed is that those students late to class missed the clicker questions. I will in future start the clicker questions later in the lecture."

Apparently, this was not carried out and the quizzes were mostly given at the beginning of the lecture. In response to the interview question "Did you like the fact that he give quizzes right at the beginning"? One student said:

Reuben: I think just being quizzed at the start of each lecture gets your brain going I think which is good except for the occasions where I missed the bus or something and I would get to the lecture a little bit late and then I would actually miss half the quizzes. That happened a few times and so that was annoying. I think probably actually for me doing it at the end might actually be better.

A remedy based on our model would be, not to start the class with assessing students' knowledge but to provoke thoughts through QDI questions. Clearly, students enjoyed their involvement in the lecture by using CRS:

Jordan: It's like if you got clicker you think this lecture more interesting because

you've got something to do, not like other lecturer sit there listen and write stuff. It's like more attention to the class.

Sara: ...so we don't just sit there and do nothing. As we normally do in other lectures. Yeah that actually help us to revise what we study.

Cathy: Fun because I think learning it has to be fun to memorise the material is really boring and long. But I'm a person into pictures. I'm quite active so I find fun activities from learning is more useful than just a dull old system.

Emily: I think it would just be coming everyday to sit in a lecture and listen pretty much, not really interacting with a lecturer, because that's what I do in my other lectures. We just go and we sit and we listen and it can be a bit tiring at times.

As part of the lectures orientation students were encouraged to read material before hand.

Lecturer: Today we covered the tactile group effort to simulate from a posterior binomial model. The clicker questions related to reading exercises I had assigned on Tuesday... They did well however some still are not doing their readings and are waiting till class and relying on last minute look-ups in the workbook – which I allow.

One student describes how she copes with her class preparations:

Cathy: Like every time I answered the questions I would flip through my course book and read immediately and to understand the questions. It was like speed-reading and learning and after the time was over Wayne would go through the answers and like immediately correct you on your mistakes so it was cool.

Students' attitude towards the use of clickers

The barplot (see Figure 4) summarises students (17) responses to the 10 questions posed. The dark colour represents strong disagreement and the lightest colour strong agreement. High strong agreement can be found in Questions 1-7 and 10 where there were no disagreements expressed at all. This affirms the use of clickers as a worthwhile feature of the class in terms of their use to facilitate immediacy of feedback; discussion of questions; comparative status with the rest of the class and anonymity. Strong disagreements were expressed in questions 8 and 9. Here students disagreed that the clickers used up too much time and were hard to use, with some agreement on both these questions. Question 10 asked whether overall they found the clickers helpful in learning the course, this was given a resounding strongly agree and agreement with no neutrals or disagreements.

Conclusion

Over the past two years, although, the basic technological instructions for use of clickers were in place by the university, there was neither any consideration for the pedagogical aspects of this new tool nor any follow-ups. The case study presented here was conducted during this period while the authors were still experimenting with appropriate ways of teaching with clickers and overcoming the many challenges that the new technology was offering.

In many respects the idea of teaching a new paradigm of statistics is a challenging thought. There are many competing difficulties to overcome which need to be addressed carefully and clarified dynamically as new material is taught. Classroom response systems played an important roll in updating the lecturers' knowledge of how students were progressing as they actively diagnosed areas of Bayesian

philosophy and thought through appropriate questions. However, the authors believe that the technology has greater potential and we can all benefit from better research based ways of implementing the technology. The construction of a well organised model was one step toward this. The authors were always fascinated with the idea of contingent teaching and the ability to take the class through various routes based on students' needs while fulfilling the goal of each lecture. Schoenfeld's [1] interactive routine for teaching is a well thought out model that can act as a base for facilitating prepared lecture material for contingent teaching, combining group discussion with accurate summary feedback and allowing in-the-moment [2] decision making. Good decision making while teaching is not easy. We expect appropriate use of CRS together with teacher's preparation in advance to be of great help in making more accurate decisions and ultimately help improve teaching.

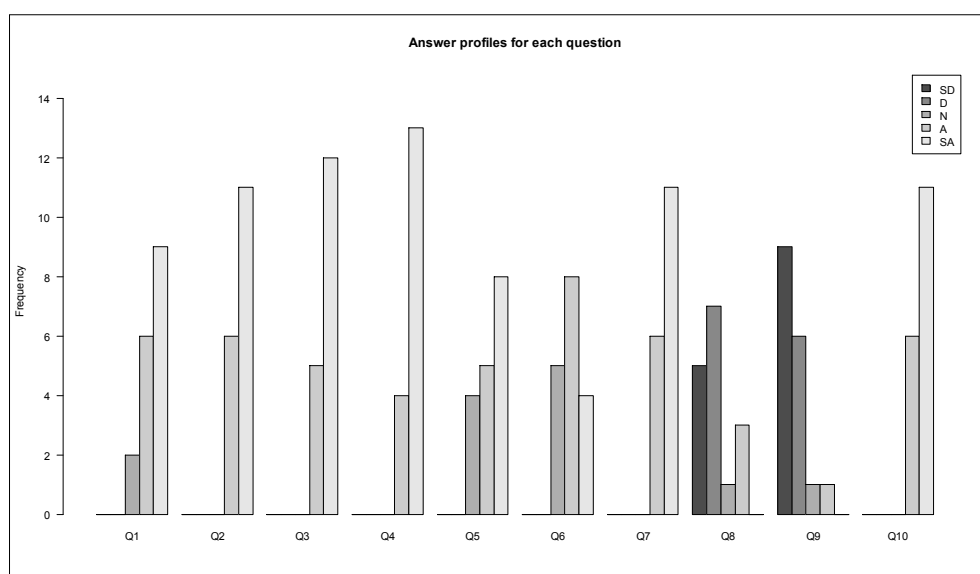


Figure 4. Bar plot of the result of the questionnaire.

In the case study described here the lecturer did not carefully integrate clicker questions in his lecture plans. The questions were often added on to an existing lesson based on the lecturers' knowledge, beliefs and goals. To address students' misconceptions he had to rely on immediate thoughts and resources at hand. It would have been far more effective to have pre-prepared material that would give a more comprehensive response if needed. Certainly, the usefulness of the model becomes more apparent as the shortcomings were revealed from the case study.

Further research is in place to examine the current climate of the clicker usage across this university. This will be accomplished by interviewing all the lecturers who use clickers in their lectures; giving a questionnaire to a random sample of lecturers who do not use clickers; sending questionnaires to students who use clickers and interviewing a sample of them. We will also trial the model in the second named author's Bayesian statistics course. Hence, the authors will prepare extra material and routes for contingent teaching with hyperlinked PowerPoint slides; QDI and clicker questions for each concept and follow the proposed routes in the model. The trial will also take place at a high school in Auckland with two teachers who use CRS in their mathematics classes. We expect the model to benefit a novice as well as an experienced teacher. The central research question would be: What is the effectiveness of this model in teaching? Due in part to anonymity of clicker use and more importantly the proposed model, we anticipate that the results of this study will

increase students' confidence to further engage with their learning.

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Teaching Bayesian Statistics: Making And Breaking Traditions

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This paper describes a lecturer's approach to teaching Bayesian statistics to students who were only exposed to the classical paradigm. The study shows how the lecturer extended himself by making use of ventriloquist dolls to grab hold of students' attention and embed important ideas in revealing the differences between the Bayesian and classical paradigms.

Keywords: ventriloquist dolls; Bayesian paradigm; classical statistics.

Introduction

Although Bayesian statistics appear in many diverse fields [1], most undergraduate courses are still taught within a classical framework [2,3]. As Bernardo [4, p. 1] declares, "hard core statistical journals carry today a sizable proportion of Bayesian papers, but this does not yet translate into comparable changes in the teaching habits at universities" [4, p. 1]. Some statisticians would argue that:

- (i) Bayesian statistics is described as subjective, and thus inappropriate for scientific research, and (ii) students must learn the dominant frequentist paradigm, and it is not possible to integrate both paradigms into a coherent, understandable course [4, p.1].

Despite the present dominance of classical statistics, some universities offer undergraduate Bayesian courses. For example Bolstad [3] has developed an undergraduate course covering the same topics that would be in a frequentist introductory course with the inference from a Bayesian perspective. However, introducing Bayesian statistics to students with a classical background is challenging. To teach the Bayesian paradigm effectively, one must take careful consideration of students' prior statistical knowledge. To prevent confusion it would be helpful to clarify the differences between the paradigms and teach both of them simultaneously. A mere repetition of classical statistics may not be as beneficial to students who believe they have sufficient understanding already.

In this paper we will describe a university lecturer's experience, implementing the novel idea of ventriloquism in teaching a Bayesian statistics course. The aim is to examine the value of using the dolls in teaching the Bayesian paradigm. We will discuss the way the lecturer brought his teaching alive by using the dolls in order to encourage students to think in line with Bayesian statistics and to clarify both paradigms.

Bayesian Paradigm

As a fellow of the royal society in 1742, Thomas Bayes was well known for his work on infinite series and was engaged in reviewing papers for a network of mathematicians such as John Canton, Thomas Simpson and others [5]. However, it was his work titled "*An Essay Towards Solving a Problem in the Doctrine of Chances*", published by his friend Richard Price two years after his death in 1763, that made him famous.

What is Bayesian statistics?

Have you ever tried driving in a foreign country where the road code demands that cars use the other side of the road? How do we overcome the prior conditioning that tempts us to drive on the wrong side of the road? Should we drive only in isolated areas where there are few cars and battle with our prior beliefs or drive where there are plentiful cars? Bayesian theory would tell us to dominate our prior with an informative likelihood that is drive where roads are busier. Technically speaking the chief cornerstone to the paradigm is Bayes' theorem which sets the posterior proportional to the prior times the likelihood as follows:

$$p(\theta|x) \propto p(\theta)f(x|\theta)$$

The *Prior* $p(\theta)$ is usually a probability density function expressing prior beliefs about the parameter of interest. As its name suggests it is what the modeller believes about the parameter prior to incorporating information from the data. The *likelihood* $f(x|\theta)$ is the way information from the data x enters the Bayesian update formula which combines the likelihood with the prior to form the posterior. The *posterior* $p(\theta|x)$ is in this way simply an updated prior. The main differences between the paradigm and the classical archetype are as follows:

- In the Bayesian paradigm probability is subjective belief whereas in the classical view probability is relative frequency.
- The object of the exercise is to update prior information about the parameter with experimental data (through the likelihood) to form the posterior. This represents the most up to date information on the parameter.
- All analyses require a prior which is one's prior subjective belief about the parameter.
- The Bayesian paradigm $\theta|x$ conditions on the data x , whereas the classical viewpoint $x|\theta$ conditions on the fixed and unknown parameter θ . This will become important when making inference on θ since classical p-values use tail areas (unobserved data) as evidence against null hypotheses.

Theoretical Framework

Teaching Bayesian statistics is a challenging task. Students arriving at the Bayesian class are often well exposed to classical statistics. However, many of them have a poor understanding of the basic measures despite their pervasiveness, frequency of use and the students' mechanical numeracy in this particular area. They will likely be able to produce these measures, but their precise meaning may still be vague to them. The P-value, confidence interval and sampling distribution are good examples of this. Certainly, some methods and tools will transfer across into the new paradigm and others will not. Tall [6, p. 6] believes that "personal development builds on experiences that the individual has met before", he defines a *met-before* to be "a current mental facility based on specific prior experiences of the individual". The difficulty for the teacher lies in deciding how best to facilitate the students' learning so that the new paradigm is not confused as a method of classical statistics rather than appreciating it as a whole new way of doing statistics. Thus, the met-befores which would hinder understanding the Bayesian paradigm must be carefully explained and differentiated to produce clarity. For example the idea of confidence intervals (a met-before) and Bayesian credible intervals, which are both intervals but have different interpretations. Tall [6, p. 6] also points out that:

The brain changes in its ability to think over time, reorganising information to create

new structures that are often more sophisticated and better at coping with new situations. It is not simply a repository of earlier experiences adding new information to old; it re-formulates old information in new ways, changing how we think as we grow more mature.

In learning the new paradigm it is tempting to carry familiar interpretations from the classical to the Bayesian context. Skemp [7] expresses these temptations as pointers toward an anti-goal, by which he means a state that one wants to avoid. Therefore, students may have a contrary emotional state in regards to the Bayesian paradigm, which can be remedied in part by reinforcing the correct emotional state.

As Zull [8] in his book, *the art of changing the brain* describes, even in our everyday language we can find metaphors that suggest “a connection between the body, feelings and thinking” (p. 72). For example, “my gut tells me or the proof is not satisfying”(p. 73). Apparently, our “emotions influence our thinking more than our thinking influences our emotion” [8, p. 74]. Mind and emotions are not separate; emotions, thinking, and learning are all linked. “What we feel is what’s real - even if only to us and no one else. Emotions organize and create our reality” [9, p. 68]. How can teachers influence the learners’ emotional state? In Jensen’s [9] view there are ways we can trigger appropriate emotions, for example: guest speakers, poetry readings, story telling sessions, debates, dramatic performances. One effective way would be “engineered controversy. Setting up a controversy could involve a debate, a dialogue, or an argument. Any time you’ve got two sides, a vested interest, and the means to express opinions, you get action!”(p. 79). Bayesian statistics is controversial and historically at the center of many debates. Naturally, there are some things that words cannot quite express and must be performed to gain better understanding. This is the heart of the Bruner’s [10] enactive representation (e.g. learning a sport or driving a car). According to his theory there are three ways to make experience a model of the world: action, visual or other sensory information and language. These forms of representation are enactive, iconic and symbolic respectively. Teaching the new perspective by exploiting debates can create favorable emotional states. The student at the tertiary level will not in general be expected to create dramatic performances. However, the teacher using a dramatic device can recreate the debate organically and in this way experience the action and in turn convey it to the students. As Skemp [7] points out: “novelty, and the distinctive quality of emotional signals, both have the effect of calling conscious attention where it is needed” (p. 19).

Method



The action research described here was in the context of a third year introductory Bayesian statistics course and was carried out at the University of Auckland in 2010. The course was fairly new and taught for the second time by the first-named author referred here as ‘the lecturer’. The lecturer together with a mathematics education researcher (the second-named author) were interested to examine the effects of his teaching style in introducing the new paradigm. Currently, this course is the only undergraduate Bayesian statistics paper offered by the department and was proposed and developed in 2009 by the lecturer to fulfil a need and to give undergraduates the opportunity to be exposed to the *other* major paradigm of statistics before they leave university. There were 100 students (mainly statistics major) enrolled in this course.

To introduce the Bayesian way of thinking and distinguish the two paradigms the lecturer used ventriloquist dolls (a few times) during the course. These episodes were short and often involved two characters. Thomas Bayes represented the Bayesian way of thinking, and Freaky the frequentist, held the classical point of view, and they

debated on some statistical controversy in front of the whole class (see Table 1 for their pictures and roles). Thomas Bayes was unique in a way that it was custom made using an actual picture of Reverent Bayes. In order to maintain novelty the lecturer only used the dolls three to four times in this course. In the lecturers' view they worked best when they were a surprise.

The data for this study comes from an interview with the lecturer conducted by the researcher at the end of the course on why using the dolls are ideal ways of teaching a new model specially to students whose minds have been set on thinking in a certain way. In addition, the lecturer wrote a reflection on his teaching and made observations about students' attitude towards the dolls. There were also regular meetings where the researcher was able to contribute by bringing the educational theories to describe the situations.

Table 1. The Dolls and Their Roles

Names	Specific Roles
Freaky 	Freaky is a die-hard frequentist statistician. He is absolutely intolerant of change and will fight any evidence that might make classical statistics look inferior. However, he thoroughly understands the classical approach to statistics and is not a fool. He is very capable and can at times expose inconsistencies that Bayesians may attempt to bring forward in debates.
Thomas Bayes 	Thomas Bayes is a very considered Presbyterian minister of the 18 th century. He is an accomplished mathematician and is not one to get excited. He is a master at presenting syllogistic logic and can find the most appropriate question to ask his interlocutor. He is smart and represents the Bayesian paradigm well.

Results

In this section we will highlight the key ideas of how using the media of ventriloquism may be an effective tool in teaching. We will explain the lecturer's beliefs on including dolls in teaching sections of the course. This is followed by two specific episodes where the dolls were integrated in teaching Bayesian statistics. Moreover, some observations from the lecturer regarding students' attitude will be discussed.

Lecturer's prior beliefs in using the dolls in teaching

Nowadays with advanced software and technologies available to enhance teaching, using ventriloquism seems to be old-fashioned and far from the setting of the modern lecture. Hence, one would be curious to know the lecturer's motivation in implementing them in his lectures. The following are some insights given by the lecturer on how it all began:

The dolls were the result of an experience I had at university while studying near a tea room. An argument broke out between two individuals and as time progressed the tone became more energetic and the volume higher. It was clear that everyone around me were listening while pretending not to eavesdrop. The impact of the argument was huge and took all of us away from the task we were doing and brought us to focus on

waiting for the response of each person. It was then that I thought of how I could reproduce this same effect in the classroom.

To implement his ideas, with no formal training, the lecturer obtained several ventriloquist dolls and incorporated them in teaching statistics courses. Freaky and Thomas Bayes were the two characters that were created specifically to help clarify controversial ideas in the Bayesian course. Freaky, though a fictional character, was portrayed to students as a voice from their past (the frequentist view) and was there to clarify students' met-befores. However, the historical character Thomas Bayes was there to promote Bayesian thinking. As part of their roles they were also used to highlight important issues by making the concepts more appealing and interesting to students.

Dolls in action

In this section, details of two teaching episodes on issues relating to the likelihood, conditionality and stopping rule principles will be discussed.

A teaching scenario involving the lecturer, Thomas Bayes and Freaky

One of the main features of using the dolls for the lecturer was to have multiple personalities in the lecture. He believed:

Dolls are a way that I could have different personalities in the classroom under my control and fashioned to have particular roles to emphasise and teach concepts that would otherwise be difficult or even boring to students. With dolls you can be virtually anything or anyone you wish.

As Bentley [11, p. 7] states, “by bringing puppets or other toys to life, you suddenly have the ability to do something bigger and better than you could do on your own.”

A teaching clip involving the lecturer and the two dolls is presented below. To have a better understanding of this debate we first present a summary of the stopping rule principle. Should inference about the unknown parameter depend on the particular stopping rule used to generate the data? Suppose an experimenter is interested in establishing the probability of a head for a particular coin. In the first experiment the coin is tossed 12 times (that is the number of trials is pre-fixed to $n=12$) this is the *stopping rule*. This experiment results in nine heads (successes) and three tails with a binomial likelihood. In the second experiment on the same coin the number of trials is not fixed, and the new stopping rule is that on the third tail the experiment ends. This scenario defines a negative binomial likelihood. However, the second experiment produces the same data, nine heads and three tails (see Figure 1).

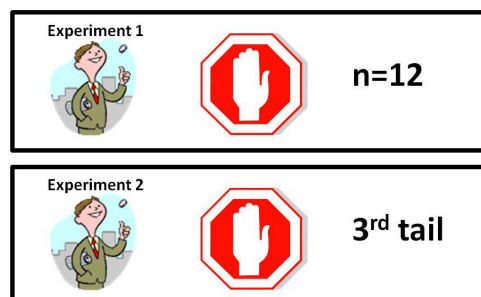


Figure 1. The stopping rule experiments.

Having controversial ideas in a Bayesian course makes it a perfect platform for dolls to have a debate in front of the whole class. In this teaching scenario the classical paradigm gives two different conclusions since the result of the paradigm's inference is dependent on the *stopping rule*. However, the Bayesian approach gives a result that is independent of the stopping rule. The following debate (see Table 1) is essentially about how the two paradigms differ and whether the stopping rule should influence the inference about the probability of a head. From the Bayesian viewpoint the stopping rule should not matter. Other issues are brought up by Thomas about how p-values break the likelihood and conditionality principles. This debate reinforces the classical paradigm and clarifies the differences between the paradigms.

Table1. An Interaction Between the Lecturer, Thomas Bayes and Freaky

1. LECTURER: Mr Freaky as you can see there are two experiments before us one where the coin is tossed 12 times and the other where the coin is tossed until the third tail, both resulting in 9 heads and 3 tails. Could you comment on these experiments?
2. FREAKY: Clearly these are two separate experiments despite the fact that they both supply the same data. What separates these two experiments is the stopping rule employed in each.
3. THOMAS BAYES: Yes, this defines the precise form of the likelihood but not its kernel. Experiment1 is a binomial, experiment 2 is a negative binomial the likelihoods are proportional to each other. We have exactly the same data and should expect the same inference in this case.
4. FREAKY: The stopping rule is a part of the experimental design and must be considered when making inference about θ .
5. THOMAS BAYES: Well at least you are consistent with your paradigm of $x|\theta$, but aren't we supposed to be making inference about θ given the data x , $(\theta|x)$?
6. FREAKY: It makes no sense to put a probability on θ , if you do then the only way forward is Bayes' theorem and this will take us into a subjective paradigm which will be of little use to science.
7. THOMAS BAYES: Bayes' theorem is already used in science and will likely be used even by yourself.
8. LECTURER: Now gentlemen let us get back to the two experiments. Mr Freaky could you tell us what would be the inference one would obtain in each experiment when testing the null hypothesis: $H_0: \theta = 0.5$ against $H_a: \theta > 0.5$?
9. FREAKY: Certainly, in the case of the binomial experiment $P(X \geq 9|H_0) = 0.073$ and for the negative binomial $P(X \geq 9|H_0) = 0.033$. This would mean at the 5% level of significance we would retain the null for experiment 1 and reject the null in the case of the negative binomial.
10. LECTURER: Mr. Bayes how would you proceed?
11. THOMAS BAYES: In both cases the posterior probability of the null is $P(H_0|x) = 0.268$. The Bayesian analysis does not use a P-value, which is in effect using unobserved data to make inference. Rather it conditions on the data we have actually obtained thus is in accord with the likelihood and conditionality principles.
12. FREAKY: This relies on a subjective prior and I don't believe in your principles.
13. THOMAS BAYES: Yes the prior used here is $P(H_0) = 0.5$, $P(H_a) = 0.5$ in both experiments.
14. FREAKY: I could suggest a different prior and then your results would be different.

15. THOMAS BAYES: Yes, given the same prior (whatever it be) the Bayesian procedures would give the same inference for θ and would not depend on the stopping rules used here. Whereas the frequentist analyses depend on p-values which in turn depend on the stopping rules. Unobserved x values are used in making inference on the parameter breaking both the likelihood and conditionality principles.

A philosophical debate on confidence intervals between Thomas Bayes and Freaky

The concepts surrounding the confidence interval are often difficult to understand [12]. The first major difficulty is that the interval endpoints once calculated for data at hand, are not random. This means that probabilistic statements related to such confidence intervals are not valid within the frequentist paradigm. The second problem is the meaning of the word “confidence” which has a strict statistical meaning in terms of repeated experimentation.

The lecturer was interested in novel teaching approaches, rather than repeating what students have already been taught in classical statistics courses. Debating the issues related to confidence intervals and bringing out the differences between the paradigms was something worthwhile to try. In the lecturer’s view:

In the case of philosophical debates on important issues, the dolls can interact with each other and bring to life something which could be flat and less appealing. The personalities can be embodied and put under your direct control. This means that you can re-make historical debates and events or philosophical paradigms by bringing personalities alive in the dolls. The dolls can represent real personalities of the present or past or they can personify ideas or paradigms. They could even be futuristic and take the students for a drive into a possible future world. They enable you to take control of a debate while appearing aloof.

Although humorous, the following debate about confidence intervals raises important questions about how to interpret them. The issues relate mainly to the interpretation of probability and open the way for the suggestion that a more useful interpretation would be that probability is subjective. The classical confidence interval is only to be interpreted in terms of the long range distribution of intervals for potential data.

Table 2. A Humorous and Informative Dialogue Between Thomas Bayes and Freaky

1. THOMAS BAYES: Ah excuse me Prof. Freaky I have a question for you. Could you tell me what a 95% confidence interval is?
2. FREAKY: Typical of Mr Bayes coming in here among us classical statisticians feigning to ask legitimate questions and pretending that we don’t know what we are talking about. We all know what confidence intervals are don’t we ladies and gentleman?! We have been using and teaching them for years.
3. THOMAS BAYES: I just want to explore your understanding of what the confidence interval is. Tell me is θ in the interval?
4. FREAKY: Who cares, we don’t know. It could be here there or anywhere, this is one of many possible intervals.
5. THOMAS BAYES: Is it likely in there?
6. FREAKY: Well of course it is likely in there. It’s a 95% confidence interval! Man !! Some people!
7. THOMAS BAYES: Is it probably in there?

8. FREAKY: Why you tricky deceptive professor Bayes. You came in here to trap and confuse me with your cunning. The parameter is fixed and unknown and the interval is fixed.
9. THOMAS BAYES: And so we have the conundrum that θ likely lies in the interval with no probability at all. What has happened since I have been away?
10. FREAKY: See that's what I'm talking about. Any opportunity to make us classicals look silly and he will jump all over it. Sitting there looking all holier than thou!



Figure 2. Thomas Bayes and Freaky.

Student's reactions to the dolls

Puppets are commonly used in schools with young children. Studies by Simon et al., [13, p. 1243] showed that as a consequence of using puppets in a science class, “the children listened more and became more involved; they were more forthcoming and able to explain their ideas better; they paid attention more quickly and became interested in the lesson immediately; and more children engaged in conversations”. Although the reports with children are mainly positive, it is hard to imagine what reactions of adults at a university level would be. Most lecturers would not even dream of doing this in front of the university students. Here are some observations made by the lecturer:

Students generally like them. They capture the imagination of both young and old and both genders. Many of my students want to see more of them and comment about this in student evaluations. On some occasions I have found students express fear of them (usually with laughter), this is because of their creepy and semi morbid appearance. With university students this is not a great obstacle although it could be for some pre-schoolers. I have had students so interested in them that they look for them and comment on their disappointment when they are not a part of the lesson. Their use does spark a tremendous amount of imagination and comment. The fact that I am not a professional ventriloquist is not at all an impediment to their use. In fact a lot more lee-way is given to me because I do not practice the art for money. The students will be a lot more forgiving to you because they see the effort and trouble you have gone to in order to help them with their education. Getting away from personal fear and using your natural exuberance, generosity and finally vulnerability will be much appreciated by students. Students do not forget well thought out doll interactions. The better the routine the more effect the performance will have.

Conclusion and Discussion

This paper is an account of a Bayesian statistics lecturer involved in action research to examine the role of dolls in his teaching. It was as part of the lecturer's aspiration to convert students' anti-goals to goals. In order to achieve this, appropriate emotional states were created through the use of dolls to stimulate students' emotional

brain and help them with learning and capturing important concepts. Studies show that the human body releases special chemicals during movement and fun activities. “These chemicals enhance long-term memory when administered either before or after learning” [9, p. 79]. The dolls were naturally entertaining and had the advantage of being semi-real, dynamic, as well as totally under the teacher’s control. They were versatile and were used to help teach many concepts. They helped to build a positive learning environment with all the elements of a dynamic debate: surprise, laughter, excitement, anticipation and interest. These positive emotional pointers came freely with little extra work from the lecturer. Whereas otherwise to construct such environment it would require an enormous amount of effort. With reference to Bruner’s philosophy, the lecturer was enacting different personalities through the dolls. Each personality allowed the lecturer to step into a different level of understanding and act the role of a person who was on that level. He had the freedom of setting the scene as high or low as he wished to help students from all stages of understanding. By using the idiosyncrasy of the person imitated he injected humour, conflict, shame, embarrassment, disappointment and many more emotions to keep the students engaged. Students’ met-befores that did not conform to the new paradigm had to be corrected immediately and philosophically discussed. It was not in students’ best interest to carry their classical misconceptions into the Bayesian paradigm. Freaky often highlighted inappropriate met-befores and was corrected by the lecturer or Thomas Bayes. The overall results were positive in the sense that the dolls captured students’ attention; brought history alive; amplified the differences between the paradigms; clarified their met-befores; contextualised difficult ideas; stimulated their minds and in the process gave students a fun experience to remember for the rest of their lives. The lecturer was pleased with this experiment and effort and intends to make further use of the dolls and fully integrate them in his teaching by making better dialogues and assessment questions relating to the sections of the debates. For professional development purposes the dolls enabled the lecturer to further his ambitions to become a better teacher. The association with the mathematics education colleague and reflecting on education theories proved to be useful as well. As Wood and Harding [14, p. 939] point out, “good teaching is not easy to identify and measure”. It is easier to show one is a good researcher by the number of quality articles produced, than showing one is a good lecturer. They suggest one way of assessing your teaching would be to perform a self review and for professional development purposes obtain peer observation.

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Teaching Differential Equations In Undergraduate Mathematics: Technology Issues For Service Courses

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This paper reviews the research on CAS use in undergraduate service mathematics courses with a focus on differential equations courses and discusses a pilot study motivated by the new requirements on CAS use in secondary school courses in Victoria and Western Australia and opportunities for change presented in previous undergraduate service courses and their assessment. The research focuses on ways to use CAS in teaching a traditional DE course, and the response of students given this opportunity in an examination.

Keywords: CAS; technology; differential equations

Introduction

Use of computer algebra systems (CAS) in mathematics courses provokes serious reflection on the way we teach mathematics and whether we need to focus on the same topics as previously. Ironically using CAS to reduce algebraic manipulation is one of its least exploited areas in teaching and it is hard to imagine reducing the time spent on the basic algebra as past comment on the poor level of entry student algebra has motivated a plethora of ‘catch up’ classes for service course mathematics in recent years. This was a particular issue for engineering courses which have substantial mathematics component. Reports of declining algebra skills also predate any school use of CAS. Using CAS in calculus however generates serious issues in pedagogy as a large amount of time is spent on a variety of specialised techniques – especially in antidifferentiation- time that may be more usefully employed on applications and modelling (Tobin, 1998). The different aims of mathematics courses need to be addressed too. A student in a mathematics major which will hopefully lead to postgraduate study and the further development of the field needs a substantially broader and deeper background than a student undertaking mathematics study to service needs in another field such as engineering. It is inappropriate to just start from the same point and merely do less – we need to frame the course in terms of outcomes.

Undergraduate courses already suffer from time pressure to provide depth and breadth of topics and give opportunity for applications which largely motivate the procedures. This is even more apparent in service courses however where the students are not training to be mathematicians *per se* but to be able to use a range of mathematical techniques which serve their specific needs. In this paper we will briefly review the literature on CAS use at undergraduate level with a focus on use in teaching differential equations and examine amended assessment approaches which incorporate CAS.

Motivation for Using CAS

Service courses such as engineering maths have suffered repeated reduction over

time as their parent courses make increasing demand on a limited time resource. This issue is observed by Buteau et al (2010) as one motive for increasing use of CAS technology in tertiary courses. They also remark on the disparity of attention given to the research on CAS use in the tertiary sector compared with the secondary sector and note that much of the work on math education in the tertiary field comes from mathematicians rather than mathematics education experts. There is clearly tension between these two groups both on curriculum in general and all calculator use including CAS in particular. The reduced maths entry standards demanded for engineering courses in the UK is remarked in Kent and Noss (2003) and this has been of continuing concern in Australia. This feeds concerns about university course content. However the belief that the issue of what should be in these types of service courses was resolved by that report by Kent and Noss (2003) is misplaced. Five years later a gathering of engineering math educators at RMIT in Melbourne could still not decide on what should be retained and how it should be taught and the issue remains open.

Many papers have addressed the issue of using CAS to enrich courses in general terms for mathematics (e.g. Pierce and Stacey, 2004) or calculus specifically (e.g. Dimiceli et al, 2010; Bloom et al, 2001) but in this paper we will focus specifically on the pedagogy associated with differential equations in undergraduate mathematics courses. This was stated by Beaudin and Picard (2010) as one topic in which CAS has had significant impact on their undergraduate calculus course. They reflect on a decade of experience in using CAS at undergraduate course level noting two key points: surprisingly little change has occurred overall at syllabus level and that CAS will not go away and should be used for maximum benefit. Puga (2001) also describes using Derive in a first year DE course in a Mexican university.

Students enter university courses with very different backgrounds in calculator usage. Wilson and Naiman (2004) suggested more extensive calculator use in schooling can have a negative impact on university performance even in a service course. Their survey result was directed at scientific calculator usage and its impact on arithmetic skills. Tunis (2004) responded to this work by Wilson and its results with a recommendation that technology be used where it enriches the student mathematical experience. That reply outlined several issues that could be addressed in the use of technology and drew a very sharp rejoinder from Wilson (2005) which missed the point of Tunis's suggestions. Wilson seemed to suggest the technology supporters were against all aspects of the traditional syllabus. This is not implied in Tunis's comments and is not the stated position of most advocates of CAS. Connors and Snook (2001) describe incorporation of CAS into a course with encouraging results and make particular emphasis on the need to maintain a balance on the use of technology in teaching mathematics and this view is echoed in Jones (2008). The interchange underlines the strong objection that this technology use elicits in traditional mathematics schools - at least at assessment level! Despite the use of CAS at school level and in final examinations in some states, it is banned in tests and examinations at most universities in Australia. Students in Victorian and Western Australian schools are expected to be familiar with use of CAS from their school mathematics in 2010 and so the undergraduates of 2011 on will be experienced in CAS and non-CAS usage in solving problems in calculus. The decision by many university mathematics departments to ban or ignore the software in courses and especially in all test assessment seems inappropriate, but doubtless it will still be used in assignment work and actual mathematical work.

There has been some research undertaken at tertiary level as well as in the

secondary classroom on affective issues associated with the CAS calculators. Surveys (see e.g Connor and Snook, 2001) show students enjoy using CAS in class and for assessment despite frequently not using them effectively or when available! Stewart and Thomas (2005) discuss student perceptions of use arising from a survey where major objections raised by students were cost and equity! These objections don't address calculator capabilities or use. This introductory use was imposed on a traditional course in a piecemeal manner with inconsistent results, which is natural in an initial exploration.

Monaghan (2007) discussed CAS use in context of the associated educational psychology and appropriate ways to graft the technology onto extant mathematics courses – when to use it as a black box and how to use it to explore concepts and enhance mathematical learning. Monaghan is concerned the technology may hamper conceptual development in some cases and argues that the French CAS research he is largely surveying may not transfer well to other education systems. This is rebutted by Blume (2007) in a reasoned response but more work is needed to resolve the dispute.

Puga (2001) discusses the role of CAS both as black box and as a tool to improve understanding of concepts and solutions in the specific process of solving differential equations. Berger (2009) describes the results of teaching a class with CAS in terms of the semiotics of the problem – the capacity of students to deal with the solution of problems in terms of the CAS sign structure and their interpretation of output. One of the key features that Puga sees as useful in CAS is the possibility to spend more time on the solutions and their interpretation and Berger's results bear directly on this issue. Berger found more than half of the class did not interpret output appropriately, which would certainly affect use of the output solutions to further explore a problem!

The actual forms of CAS vary from specially designed overreaching packages like Mathematica (see Berger, 2009) and Maple (see Jones, 2008) to limited free software like Wolfram Alpha (Dimiceli et al, 2010) and the versions embedded on the more advanced graphics calculators which have also arisen in cases like the TI set from a previous computer package – in that case Derive (Puga, 2001). Most universities encourage the use of Mathematica or Maple which are suited to regular use outside of test assessment but they have generally frowned on CAS calculators which give more limited but very portable access to CAS for all students. Jones (2008) uses the CAS package Maple TA for an online assessment tool in assisting the teaching of mathematics at undergraduate level. We also examine ways to structure questions to bring out features we wish to retain and this creation of multiple part questions in the DE area is of particular interest to the authors.

Pedagogy of Differential Equations

The key issues prompted by this technology are how to use it effectively in assisting the teaching of mathematical concepts and how to use it in assessment. The prospect of streamlining topics to match the modern software option needs to be considered. We focus on differential equations (DEs) as provided in standard service courses. At present the topics covered frequently include many or all of the following:

- the use of differential equations in modelling problems,
- the solution of first order linear differential equations either directly or by variable separation or use of integrating factors,
- the solution of second order linear differential equations with constant coefficients by use of complementary functions and particular integrals,
- the use of power series solutions,

- Laplace transform procedures,
- coupled differential equations,
- numerical solution techniques and
- some procedures to solve partial differential equations.

These topics stretch across several years of subjects commonly and all the solution procedures in service courses underpin the use of the differential equations in practical situations from mechanics, heat transfer and fluid models to wave mechanics. Similar topics were found by Arslan (2010) looking at procedural versus conceptual learning in differential equations. Arslan found that conceptual learning from traditional courses was much poorer than procedural learning and suggested that the use of technology to boost multiple representations of solutions may assist in the teaching of concepts.

Clearly, modelling with differential equations lies outside the ambit of technology and is among the harder topics for students. Setting up a DE from given information is a higher level skill, and important for anyone using these tools. It is desirable in some service courses like engineering or economics. It would be useful to spend *more* time on this aspect. Connor and Snook (2001) give one example of a DE exam question in a spring mass system where this type of feature is the focus of the problem.

The solution of first order linear differential equations by direct means or separation of variables leads to a straightforward application of calculus. Students certainly need to see this even if they sometimes will use CAS to solve such a DE. Solution of such DEs by integrating factors is useful in itself and motivates the development of solving second order DEs by auxiliary equations. As with the first order case we may move on then to using CAS to solve problems in practice.

Classic solution of DEs has used Laplace transform procedures with tables and the associated algebraic manipulation that this entails. This seems one type of solution process asking to be replaced by technology just as direct scientific calculators allowed us to sidestep log tables and sliderules. The Laplace transform is widely used in engineering and signal processing but in some service courses use may be reduced. Solution by power series enables the development of work using special functions and is useful in outlining where solutions like Bessel functions and Legendre polynomials arise. This may be useful to retain in some but not all service courses. Coupled differential equations extend the model capability of DEs – for example they open up prey-predator models in biology or some catalytic reaction mechanisms in chemistry - but they also provide a way to introduce the way we reduce all higher order DEs to systems of first order ones.

Numerical solution procedures are naturally arithmetic and time intensive, need computing facility to do efficiently anyway, and need to be included in any course on differential equations. The solution of partial differential equations is commonly done in special cases by techniques like variable separation to reduce problems to ordinary DEs or by use of numerical procedures. In a service course the detailed analysis of these equations using the theory of partial DEs is generally ignored.

Technology use to shift the way we teach differential equations has a long history. In 1970, Bajpai et al were advocating this policy with far more primitive and less accessible technology at their disposal. However the new handheld CAS enables the technology to be immediate, easily used and available in examinations and tests. Ferzola (1994) also advocated use of CAS in teaching differential equation courses. Roubides (2004) describes an attempt to redesign a traditional DE course including technology with some student feedback on the pilot program suggesting that it was more popular than its predecessor and at least as effective as a teaching approach.

The research here concerns itself with a limited set of issues. Can we draft questions from a traditional course so that they are suited to CAS use and still bring out the features we seek? Will students use the CAS effectively or at all to do this if they are taught with both approaches? Could some students use CAS solutions to backtrack to obtain other information asked for in a question, giving a new solution path?

Using CAS in Test Problems With Simple Linear DEs

The authors examined one previous paper set to two groups - one using CAS and one without. The paper included 8 questions where CAS could be an advantage and unsurprisingly the students with CAS outperformed those without whereas on the papers overall this was not true and in fact the CAS group were ostensibly a lower mathematics entry cohort. This reminds us that where CAS is allowed the papers need to reflect this! There were several standard ‘technique-type’ DE questions given and we look at some approaches here.

Question 1

Solve $\frac{dy}{dx} = xy - 2y + 2x - 4$ for y as a function of x .

This involves solving a first order separable differential equation- this could be done with one command using CAS. The question may be better constructed as follows:

Question 1A

Show that the following DE is separable and hence or otherwise solve for y as a function of x .

$$\frac{dy}{dx} = xy - 2y + 2x - 4$$

This allows the testing of separability and still gives some chance for an answer to be found by CAS or checked by CAS.

The process of separating variables is worth 2 marks and the rest could receive 4 marks. A student giving the solution correctly by CAS would get only the last 4 marks. In practice these marks reflect student ability to use a number of features of the calculator which generate the solution – the ‘deSolve’ instruction and use of the prime symbol for differentiation, correct syntax in the instruction and correct reading of the output. Overall 6 marks can be allocated either way. A similar approach can be used to require use of an integrating factor. For second order DEs we can take a step approach as well. A typical example would be the following.

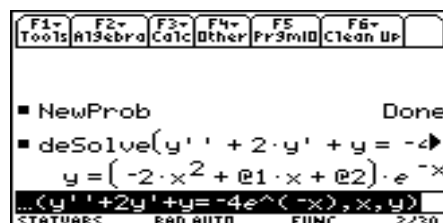
Question 2A

For the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

find and solve the auxiliary equation and solve the DE. Hence or otherwise solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = -4e^{-x}$$

In this problem solving the homogeneous DE analytically could receive 5 marks and the subsequent analytic solution of the inhomogeneous DE would obtain perhaps another 6 marks identifiable.



These latter marks would still be available to the CAS user, whose solution process with correct input would generate a screen like that shown.

There is one important issue this brings out. If we lead a student through a DE like this we are giving away part of the solution process. In doing so we signal that we require them to use a particular technique e.g. separation of variables or use of an integrating factor rather than make them identify the appropriate technique. This sacrifice may be worth making. If this identification process is seen as important it can be examined separately in a CAS-free test. In Victorian schools at final year students receive two maths papers and one is required to be calculator free. This policy is useful to retain in undergraduate courses. This can address concerns about excess reliance on CAS in inappropriate circumstances and ensure that concepts are tested in a CAS neutral forum.

Analysis of Examination Questions

The consideration of these past papers and the DE questions on them led to us framing examination questions in that form in recent work. One of the authors ran a pilot project in semester 2, 2010 where students were permitted to use a CAS calculator. This first year mathematics service subject was given to a cohort doing little mathematics and generally of lower entry level on mathematics scores. The examination included 3 questions on solving ordinary differential equations. Students were instructed to show their working in the answers but had also been informed in classes that if they did not know what to do, partial credit could still be awarded if they wrote down the answer using CAS alone without showing steps. Students had also been advised that use of CAS could help them check their work. They had also been instructed during classes in how to use the 'desolve' command to solve first order and second order differential equations, including initial value problems. There were 28 students who sat the examination.

All students had use of a CAS calculator, with the majority using the TI-89 Titanium calculator and a smaller number using the TI-NSpire. For solving first-order and second-order differential equations, the syntax required was the same for both models of calculator, so in terms of use of the calculator for this type of problem, the only small differences were the sequence of menus required to access the correct command and the different presentation of the buttons used on the calculator to type the syntax.

The relevant examination questions were split into several parts as described before. This checked students' understanding of the different steps in the processes involved in solving each type of differential equation. We consider each of these questions, and students' answers to them and then look at how students performed on these questions relative to how they performed on the rest of the paper, including how the students who used CAS performed compared to those who showed no evidence of

using CAS.

Question 5

Consider the differential equation $y'' - y = e^x$.

1. [1 mark] Write down the auxiliary equation for this differential equation.
2. [2 marks] Determine the complementary function for this differential equation.
3. [4 marks] Determine a particular integral for this differential equation. Show all working.
4. [2 marks] Write down the general solution of this differential equation.

This question was designed to test students' knowledge of the steps involved in solving a second-order linear differential equation with constant coefficients. Any students who did not understand the process at all but could use CAS to solve such a DE would be expected to write down an answer for part (d) only. It was also thought that students who did not know the full process but who understood that the solution comprised the complementary function plus the particular integral might use the CAS to get the general solution and then write down answers for (b) and (c) as well, but without showing any working and without giving any answer for part (a).

Table 1. Student Responses on Question 5

5(a) Auxiliary Equation

Result	Frequency
Correct	13
Wrong Method	11
No Attempt	4
Total	28

5(b) Complementary Function

Result	Frequency
Correct	8
Correct Consequential given wrong part(a)	3
Wrong	10
No Attempt	5
Correct General Form only	2
Total	28

5(c) Particular Integral

Result	Frequency
Correct	3
Wrong	9
No Attempt	6
Correct general form of y_p only	8
Wrong choice of y_p else OK	2
Total	28

5(d) General Solution

Result	Frequency
Correct answer only	4
Correct answer with explanation	1
Wrong	6
No Attempt	6
Correct consequential given wrong (a) or (b)	10
Correct: Method and CAS	1
Total	28

Only five of the 28 students who sat the exam demonstrated obvious use of CAS in answering part (d). Interestingly, these students had all lost marks in some of the

previous parts of the question: one of them did not attempt any other parts of the question, one obtained incorrect answers to all the other parts of the question and the other three made errors in the logic for two of the other three parts of the question. It was also interesting to note that this was a question where the output for the answer to (d) given by CAS was slightly different to the form of the answer they would have obtained using hand-calculation:

By the usual method of hand-calculation as outlined, the answer was of the form

$$y = Ae^{-x} + Be^x + \frac{1}{2}xe^x.$$

The CAS presented the answer in the form

$$y = C_1e^{-x} + \left(\frac{x}{2} + C_2 - \frac{1}{4}\right)e^x$$

and this is how all five of these students wrote their answer.

No students obtained the wrong answer to (a) and then wrote the correct answers for all of (b), (c) and (d), suggesting that while the above five students who made errors in earlier parts of the question knew how to type ‘desolve’ to get the answer to (d), it did not occur to any of them to write down or check their results for y_h and y_p in parts (b) and (c) based on this output. (Since the output from CAS differed from the way they usually set it out, this might have deterred them from trying to deduce the correct answer to the other two parts of the question.)

The standard of answers to this question was generally poor among the other 23 students as well. Two students did not attempt any parts of this question. Only two students obtained correct answers to all four parts of this question, with these two students showing working for each part and giving no evidence of use of CAS. The distribution of responses to each part of the question is shown in the tables provided.

Question 6

Consider the differential equation $3x^2y' = y^3$

- [1 mark] Show that this differential equation is separable
- [5 marks] Find the specific solution of the equation satisfying the condition $y(2) = 1$. Show all details. Write the final answer in the form $y = f(x)$.

This was designed to test students’ ability to identify when a DE is separable and how to solve it. Understanding how to deal with an initial value was also tested. We expected some students would not be able to separate the DE and would hence revert to writing down the answer from CAS, with or without the initial condition included. Only nine of the 28 students obtained a correct answer for part (a), showing that the DE was separable. One student who did not obtain the correct answer for question (a) obtained the correct answer for (b) by hand, demonstrating understanding of the general process of solving the equation. Only two students demonstrated any use of CAS in part (b) of this question. One of them had obtained the correct answer to (a), then had started calculating (b) incorrectly before resorting to CAS to obtain an implicit answer (i.e., not in the form $y = f(x)$). The other had not attempted part (a) at all and did not show any hand calculation steps: in (b) they used CAS to first solve the DE implicitly, and then again to solve the resulting algebraic expression for y .

Question 7

Consider the differential equation $xy' = y + x$.

- [2 marks] Describe (classify) this equation, providing as many appropriate terms as possible.

- b) [3 marks] Find an integrating factor for this equation.
- c) [3 marks] Hence find the general solution of the equation.

This was designed to test students' knowledge of solving a first-order, linear DE, together with being able to identify that it is indeed first order and linear. It was of interest to see how many students who clearly did not understand the general process would just write down the answer to (c) without attempting (b). It was also investigated whether students correctly identified the type of DE they were dealing with, as in part (a), to further test their general understanding of the concepts. In part (a), only 7 of the 28 students correctly identified the DE as both first-order and linear while 14 of them did not correctly identify either of these characteristics of it.

For part (b), finding the integrating factor, 8 students made no attempt at all, and only 6 students obtained a fully correct answer. For part (c), surprisingly, only 3 students demonstrated use of CAS as follows. One student showed no hand-calculation steps and their CAS answer was fine except that the constant was copied down wrongly from the output screen- this student had given the correct integrating factor in (b) and had not attempted to classify the DE in (a). Another student gave wrong answers to both (a) and (b) then clearly resorted to CAS in (c) as they simply wrote down the correct answer without showing any other working. The third student obtained (a) partially correct only, (b) wrong and then in (c) tried one line of the calculation by hand and then resorted to directly using CAS to obtain the correct answer.

The results showed proven use of CAS by only 7 students (25% of the exam cohort). This may reflect their limited experience with CAS at school level. In 2010 most students entered from courses where CAS was not used unlike in 2011 onwards. Of these students, only one used CAS for each question and one for two of the questions. Their total marks on DE questions were 4 and 8 respectively out of a total of 23, showing that mastery of the tool was not good. The five students who only used CAS on one DE question had total DE scores ranging from 1.5 to 12 out of 23.

Discussion

Across all 7 students who used CAS for at least one of the DE questions, the average score was 6.1 out of 23 (SD = 3.6). Their total exam marks ranged from 20 to 84 out of 140 with an average mark of 50.3 (SD = 21.8). Interestingly the student who obtained 84 overall was the only student to use CAS on all three DE questions. Among students who did not use CAS at all in the DE questions, the total scores over the DE questions ranged from 0 to 20.5, with an average mark of 7.2 out of 23 (SD = 6.5), while exam scores ranged from 33 to 120 with an average mark of 66.7 out of 140 (SD = 28.2).

Possibly due to the small number of participants in the study, independent groups t-tests showed no significant difference between the mean exam score or the mean total DE questions score of students using CAS on at least one DE question compared to the mean score of those who did not use CAS at all. Although there was more variation in scores for the students who did not use CAS, both on the DE questions and on the overall exam scores these differences were not found to be significant.

The total number of marks available for the three DE questions was 23 so students typically performed poorly on these questions. Comparing results for the students in this year to the equivalent group in the previous year, suggested this cohort of 28 students were relatively weak, which might also account for the relatively low scores of a fair number of these students, both in the DE questions and the exam as a whole.

Table 2. Group Statistics

	CAS	N	Mean	Std. Deviation	Std. Error Mean
Total score on DEs questions	No	21	7.2143	6.48184	1.41445
	Yes	7	6.1429	3.57904	1.35275
Total exam score	No	21	66.7381	28.15520	6.14397
	Yes	7	50.2857	21.83433	8.25260

The results give some answer to the research questions. We can draft questions in a way to elicit all the features we required in traditional courses and still give partial credit to CAS use. Results show CAS use here is patchy at best by students however (cf Stewart and Thomas, 2005) and that students have yet to use the CAS to be able to backtrack to solve aspects of a problem other than the direct DE solution and this may be due to problems of interpretation like those found by Berger (2009). The results suggest CAS imposed on a traditional course may limit efficacy so further research is needed.

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Tertiary Mathematics Learning and Performance in First Year Mathematics in the Environmental Sciences: A Case of Student Preparedness for Learning Mathematics

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Australia and New Zealand are presently experiencing a decline in student numbers in higher mathematics, both at universities and high schools even when mathematics and science skills are critical for our long term prosperity. It is then critical to address some fundamental issues affecting the learners in mathematics classrooms. Observations of high schools students and university entrants to first year mathematics show that students are not being prepared for learning itself or let alone for mathematics learning environments. In classrooms, many students fail to actively take part in learning voluntarily; or even through an “active push” by teachers to further their own learning. The non-learners tend to become a problem for teachers to manage and when teachers themselves are less mathematically prepared the problem becomes more complex. The dealing of management issues leaves less teaching time to promote mathematics or provide direction to those prepared to extend their learning. The author believes the loss in “teaching time” in math classrooms is significant. This paper addresses the issue of “student preparedness” in the first year mathematics examining factors such as self-directed approaches in learning and determination to succeed. The results show that student preparedness for learning environments is an important factor in success in first year mathematics. Preparing students for learning is a great challenge for parents, carers, early, primary and high school and tertiary institutions. Therefore, it is critical for researchers to address student preparedness variables within classroom contexts and investigate their effects in learning and performance.

Keywords: student preparedness, first year mathematics, strategies for mathematics learning environments, multivariate methods, mathematics education, PLS

Reflection on Learning Environments

In this ever more competitive global economy, Australia’s science, engineering and technology skills need to match the best in the world.

Prime Minister John Howard, speech in Sydney, 18 September 2006

Every advanced industrial country knows that falling behind in science and mathematics means falling behind in commerce and prosperity.

Gordon Brown, UK Chancellor of the Exchequer, Budget speech, March 2006

The situation concerning mathematics learning in Australia does not appear to be satisfactory given that undergraduate students continue to choose degree options away from those with significant amount of mathematics (Chinnapan et al, 2007; Hall,

2007; Hughes, 2006; Rubinstein, 2011; Wright, 2007). This situation is not only noted in the universities but also in high schools where fewer numbers of students are now undertaking math C for example (Queensland, Australia) (Hall, 2007 Wright, 2007). Chinnapan et al (2007) identified a number of difficulties concerning the learning of mathematics in Australia such as: lack of quality and qualified teachers, inadequate primary teaching of mathematics and science, lack of access to advanced high school mathematics courses, lack of appropriate university prerequisites, lack of understanding of the role of mathematics including a decline in student doing higher math. Others such as Britain, US and Ireland also appear to be battling similar issues (Hourigan, and O'Donoghue, 2007; Hong et al, 2009; www.irishexaminer.com/ireland/kfojgbsnsgb/rss2/). Clearly, there are problems in the teaching and learning of mathematics including inappropriate beliefs and attitudes towards the importance of mathematics itself, but the author believes that the causes are more closely related to student variables in "learning environments" generally; and more particularly, to the level of student self "preparedness" for learning environments (Vondracek - Chronicle of Higher Education, 2006; Sanoff, 2006; Frey, 2010).

There has been years of research on teacher related variables (Brandell et al, 2008; Harris and Jensz, 2006; Hong et al, 2009) with clear benefits but equally important is to search for answers in the assessment of the learner themselves - "student preparedness related variables"; for example, the makeup of the "modern student" needs to be understood better (Higgins et al, 2010). In class observations show some disconnect in the modern learning environments. Students tend to be rather inactive and potentially disengaged learners in classrooms. The key attributes, characteristics and behaviours during learning contexts need to be understood better and only then instructors may begin to instil long term learning skill and attributes useful for lifelong learning.

In this paper, the author explores mathematics learning in tertiary studies, particularly focusing on the student preparedness in higher mathematical studies. The levels of preparedness of students in mathematics learning environments are examined in terms of overall performance using multivariate methods. The reflective and qualitative analyses is also undertaken and is based upon observations, student interviews and indeed the author's work in the field; 20 years of tertiary lecturing and high school teaching to all levels of students over three universities. In terms of self-determination and preparedness variables, tertiary institutions appear to be important field sites where student preparedness for learning and student intrinsic motivations may be assessed. Essentially, students are expected to be self-prepared, motivated and self-directed regarding their investments of time and/or engagement; and in choosing approaches to learning environments, unlike high schools where work is more directed by others. In this manner, tertiary institutions are where one can test or at least observe, given years of early and high school learning and instruction, the levels of "preparedness". Admittedly, the tertiary student cohorts tend to be biased but increasingly more students are from diverse backgrounds (Engelbrecht and Harding, 2008). Tertiary learning in the end depends largely on student preparedness variables such as their abilities to voluntarily manage learning approaches, strategies and time; including the amount of hours invested to succeed.

Research on Learner Preparation and its Assessment

There have been rather few studies into preparedness of tertiary students in the literature (Albion et al, 2010; Bahr et al, 2004; Tonkes et al, 2009; Wilkes, 2010). The

work done mostly concerns student content preparedness or teacher preparations; such as improving the teaching of content by trialing methods of teaching; but few studies genuinely investigate student preparedness for learning environments (Byrne and Flood, 2005). Biggs (1996) states that learning approaches are not intrinsic characteristics of students but rather the learning environment influences the process; prior experiences, motives and intentions influence learning. Inherently, this suggests separate preparations for learning itself; that is, student needs to learn about learning itself and acquisition of expertise (Sternberg, 2003). Byrne and Flood (2005) found that students entered university because of a mixture of intrinsic and extrinsic goals and had positive expectations of tertiary studies. Yet around 25% lacked confidence and did not believe they would succeed in their first exams. Students expected lower levels of work commitments at tertiary level even when research shows effort invested often determines success (McInnis, 2003). Byrne and Flood (2005) suggested much further study; regarding the influence of motives, preparedness and expectations even when they agree with much of the education literature exists that show intrinsic motivations lead to deeper learning; and surface type understanding is connected to extrinsically motivated learning; more study on motives, preparedness, or learning approaches to determine which factor or factors lead to deeper learning and are related to academic performance; and to compare stated confidence levels to expected examining the importance of confidence as a factor related to abilities of students to judge their own capabilities. It appears that self-assessment capability aids in one becoming a “mature and progressing” learner (Garrigan, 1997; Yorke, 2001).

The account of the preparedness of the learners may be noted in the early work of Sternberg’s (1996) triarchic theory of intellectual abilities (TTIA, Sternberg, 1996, 2001a, 2003) and the mental self-government (TMSG, Sternberg, 1998). In his framework, thinking styles, biographical information and past achievement help predict academic performance. Sternberg argues that student learning and thinking style (Sternberg, 1997) together with ability levels are important (Sternberg & Grigorenko, 1997). TMSG is related to the nature of thinking styles providing individuals their preference of thinking patterns, while the TTIA focuses on the ability itself. In essence, TMSG refers to different thinking styles that dictate the preference in the use of abilities. Sternberg also defined the concept of “successful intelligence” as an alternative to IQ. This concept includes a number of characteristics such as whether students know their strengths and weaknesses, whether they are goal setters, whether they are highly motivated, whether they are able to follow through with promises made, whether they are high in self efficacy, whether they know the problems that needs addressing, and whether they can translate their thoughts to actions to achieve real world achievements. Many of the characteristics may be noted in student preparedness variables. The preferences of thinking styles can be guided by preparedness for learning environments and this may be in turn based on biographical information (Sternberg (2001a, b, c).

Much work has been done in the affective domain related to mathematics learning (Bandura, 2005; Byrne and Flood, 2005; Kelson and Tularam, 1998; Tularam and Kelson, 1998). The process of quantifying and monitoring affective factors or indeed assessing the nature of the role they play is not an easy task (Tularam, 1998). Mathematical self-concepts and intrinsic motivations tend to be the two main interest areas but less work appears on understanding student preparedness, willingness to study mathematics or approaches to learning when in classroom full of students; that is, whether students optimize opportunities of learning. The self-concept includes mathematics talent, confidence, self-efficacy, and anxiety; while student interest,

enjoyment, intellectual stimulation, reward for effort, valuing mathematics, diligence relate to motivational factors. Carmichael and Taylor (2005) noted factors such as confidence, motivation, and engagements are often used without clarifications. Bandura (2005) has argued for the stricter definitions of self-concept, confidence, and self-efficacy are needed. A learner may be confident within one part of the content while not in another within the same course content thus an overall measure seems meaningless. A student may have high level of self-efficacy (attribution of failure to changeable factors) for finding the derivative of a polynomial but a low level for a cubic. The assessment of student work will need extra data regarding preparedness for learning, self-determination or willingness to self-study mathematics and so on.

Different cultures may provide interesting notions of preparedness for learning. It is true that preparedness for learning can be based on beliefs, norms and values held in cultures. Kennedy (2002) says: “often researchers suggest that for example the Chinese learners have to be weaned off ‘inferior’ or ‘old-fashioned’ modes of learning onto ‘deeper’ ways of understanding. Cort[azz]i and [Jin] cautions against such cultural imperialism: ‘there is no reason to suppose that one culture of learning is superior to another . . . this needs to be kept in mind when teaching methodologies migrate around the world’ (1996: 174). In fact, a better understanding of such ‘Chinese learning styles’ as ‘deep memorization’, collaborative group learning (Tang 1996, Kember 2000) and the pastoral role teachers play outside the classroom (Pratt et al, 1999) could well benefit the Western learner” (p. 442). Clearly, other cultures could be included but analyses show students from Singapore, Hong Kong, China, China Taipei and Japan do extremely well in mathematics tests (The Condition of Education 2001); not only recalling of facts and routine problems but also in non-routine, reasoning and problem solving domains. However, this aspect can be addressed in another paper. This study examines the relationship between preparedness and overall performance by examining over the semester performances of the first year environmental undergraduate mathematics students.

Factors such as willingness to learn, developing a want to learn and more importantly, making an active decision to learn or to become a learner in a learning environment appear critical. Being prepared for learning involves active engagement in learning that in turn requires one to self-develop strategies on how to go about being a prepared student. Most students believe that the essence of studying is to “pass a test or exam”; although an important factor, it is argued that for a prepared learner this is not only what learning is about. Preparedness includes self-derived efforts to conduct concentrated and “independent study” in and outside of learning environments.

The Study - Qualitative and Quantitative Methods

This study is of a reflective nature concerning in class observations conducted over many years including a study of first year tertiary students’ performances over same time. More particularly, this study concerns the performance of first year environmental sciences majors as well as a reflective analysis on math classroom observations (3 schools) over 5 years. The total assessment includes problems sets (18%), exams (70%), and “attendance and input level marks (12%)”. The attendance and input determined by tutors who interact with the same students for 13 weeks over a semester (one to one manner). In addition to lectures, the lecturer led 6 one hour optional help sessions (workshops) per week for 13 weeks. The workshop attendance and input was recorded weekly while the lecture attendance recording was random (20 per year). The attendance to tutorials, workshops and lectures all provided an

insight into voluntary efforts made by students when in learning environments. The mid semester test (30%) examined knowledge gained during semester (6-7 weeks); while the final exam of (40%) was deemed to be a measure of overall achievement (15 -17 weeks from the start). Rather than simply exams and routine fact type problems, the assessment methods include a problem sets component sets requiring much investment of time, effort and research (2-3 weeks of work per set). A mark out of 3 was given for effort placed in the overall in each set. In addition to attendance, the tutors were asked to provide a mark for effort for each student during tutorials; and to record the number of times students sought help outside of classes. The lecturer also recorded the number of attempts a student made to seek help outside of class throughout the semester. The marks were then categorized as high, medium and low that in turn led to categorisations of student preparedness as high, medium and low.

Students were requested to develop an A4 focus learning sheet starting early (one for each exam). This sheet facilitated qualitative assessment demonstrating the level of preparedness for exams in terms of organisation of sheets, whether focus was on concepts, rules, examples, level of detail, and whether the sheet was fully used in an efficient manner etc. Student level of attention to detail, learning strategies and organisational skills could be exposed in such writing. Five years of focus sheet were available for analysis but only a few and of certain groups of students were analysed for this study: 4 sheets each from High ($x > 85\%$), Medium ($75\% < x < 50\%$), and Low ($x < 35\%$).

The dataset of almost 80-120 students yearly was also analysed also using multivariate methods to understand significant relationships; and partial least squares regression (PLS) as used to analyse the correlated data. The detailed solutions of problem sets and student attendance to lectures, workshops and tutorial related to students' preparedness: self-motivated effort, independent efforts and general preparedness for attending learning sessions. Student preparation for the mid semester was also critical for the lecturer determine sustained efforts with workshops and lectures; that is to maintain work levels to achieve. Method of proxy measures of such type described has been used in the past; for example, Bahr et al. (2004) relied on demographic variable as a rough proxy measure for academic preparedness (a synthesized college-level indicator for the average academic preparedness). Palmer et al. (2011) studied personal and dispositional characteristics, achievement motivation, conscientiousness, academic discipline and commitment to study for success at university, by means of interviews, portfolios, essays, reports and evidence of extra-curricular activities. Therefore, personal and dispositional characteristics, were examined through observations and interviews during workshops, lectures and tutorials during the semester and before after exam periods.

In particular, the part of the study investigated the influence of marks attained in the three (Problem sets, Attendance and input to tutorials, and Mid-semester) to Final/Total mark in a similar manner to Hemmings (2011), who used the multiple regression (MR) analyses showing a test state-wide test explained much of the variance in examinations. Also, Stols and Kriek, (2011) used stepwise regression to identify the critical predictors and verified the results using partial least squares. The design and method of analysis used were based on previous work (Bahr et al, 2004; Hemmings, 2011; Palmer et al, 2011; Stols and Kriek, 2011)

Results

The correlations among the assessments items are presented in Table 1. The problem set, attendance, mid semester are correlated with each other and with the

final. The stepwise MR analysis selects problem set and mid semester as the main predictors while attendance factor is eliminated due to collinearity. For this reason both variables are investigated further and the two models are significant (Table 2 and 3) suggesting much of the variance in the final score may be explained by the student preparedness variables: the problems sets included researched and detailed solutions for the attained of a higher problem set score; and self-determined efforts to attend lectures, workshops and tutorials to obtain help to gain a better understanding were related to the attend - attendance and self-motivated input variable; finally, maintaining sustained effort over weeks to succeed over the months in a semester in an effort score better in the mid semester concerned the mid variable.

Table 1: Correlations Between Assessment Variables With *P* Values (* 0.01)

	Probset	Attend	Mid	Final
Probset	1			
Attend	0.74*	1		
Mid	0.59*	0.41*	1	
Final	0.67*	0.45*	0.73*	1

Table 2: Multivariate Regression Model 1 With Problem Set and Mid Semester

Model 1	Coeff	Beta	Sig
Const	-8.744		.005
Mid	0.877	0.508	.000
Probset	1.084	0.368	.000

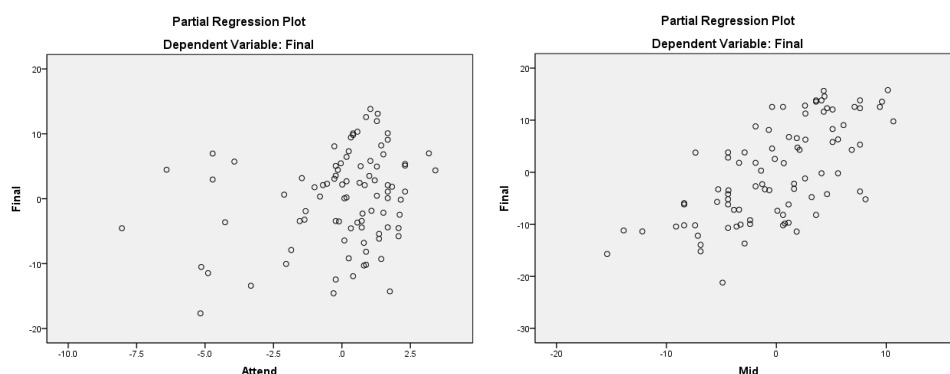
Table 3: Multivariate Regression Model 1 With Attendance and Mid Semester

Model 2	Coeff	Beta	Sig
Const	-6.435		.083
Attend	0.808	0.183	.022
Mid	1.125	0.652	.000

Table 2 shows that Model 1 involves problem set and mid semester as the main predictors while Table 3 shows Model 2 with attendance and mid semester variables being regressed with final. Both models give strong R^2 values of 0.62 and 0.56 respectively; when attend alone is regressed with final an R^2 of 0.45 was noted; almost half of the variance in the final mark explained by attendance variable.

Figure 1 shows the clear trends in each variable when partially regressed. The variance inflation factor ($VIF < 4$) confirmed that no serious collinearities in the two models. Further analysis using the partial least squares (PLS) with the final mark as the dependent led to the following model equation with significant coefficients:

$$\text{Final} = -14.32 + 1.32*\text{ProbSet} + 0.36*\text{Attend} + 0.71*\text{Mid}. \quad (1)$$



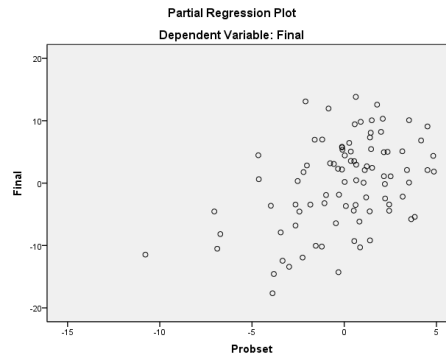


Figure 1. Partial regression plots of Probset, attend and mid with final.

Figure 2 shows about 60% of the variation was explained using PLS with final. Figure 2 shows the percentage of variance explained by each of the scores reported and model fit. The student preparedness factors were found to be significant and the model predicted the final score with some accuracy. Table 4 shows problem set explained most of the variance followed by attendance and mid semester scores.

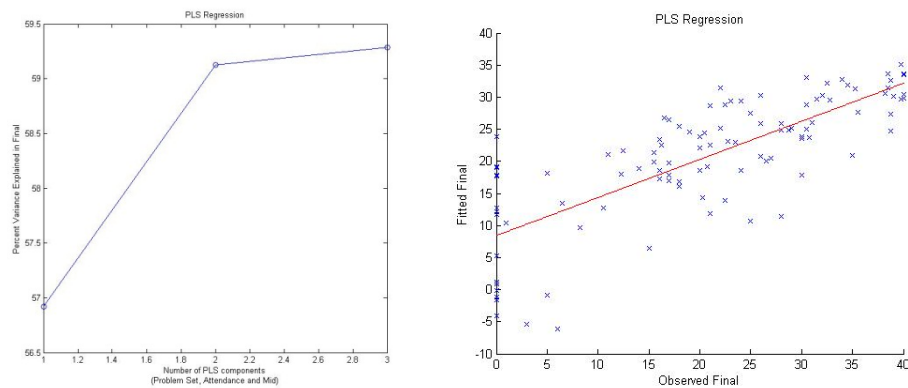


Figure 2. PLS Regression – Percentage Variance Explained and Model Fit

PLS components explained around 76%, 20% and 3% respectively with each component relating to problem set, attendance and mid semester performances in term of the final. Thus suggesting student preparedness led to higher level of strategic inputs throughout the semester.

Table 4: Percentage of Variance Explained in PLS Results

PLS Model	Percent Variance Explained		
	Problem Set	Attendance	Mid
PLS Comp	0.762	0.207	0.0308
Final	0.5692	0.022	0.00161

Summary of Qualitative Analyses

Both types of analyses highlighted the importance of problem set and attendance being strongly correlated and that problem set and mid semester were major contributors. The students who demonstrated self-direction, determination to learn, independent work attitude, high attendance, high effort and sustained work towards

mid-semester scores and final, performed well. The level of self-determination and motivation to succeed seemed to determine whether higher effort was placed throughout rather than close to assessment periods. High level of students' self-motivation and belief was noted in successful students; positive belief in ability to successfully complete and attribution to effort were important. Those who believed that they were not good at math and the work was not in their capability indeed performed lowly; and even when they passed, their mark was only a lower level pass mark; while those who believed they were lowly but capable of better appeared to strategically place their efforts to focus on their weaknesses and actively sought help; performed better than they expected. This relates to what Sternberg (1996, 2001a) labelled "successful intelligence" in that those who realised critical factors for success developed strategies to cope; the successful students knew they had to translate their strategies into actions and this was evidenced in effort, motivations, and interest developed to perform well. The high "preparedness" students were determined to plan and direct their learning and effort and this in the end led to success. In the end self-persistence and diligence helped those when had lower prior mathematics levels to experience success in many cases for the first time. It is to be noted that 20% withdrew and about the same failed in the course in 2011; the average withdrawal and failure rate is around 17%. Over the years, students who withdrew or failed this course were mostly insufficiently "prepared for learning environments".

Discussion and Conclusion

The reflections, review of the literature together with the analysis and results together show the importance of student preparedness in overall performance. The results of this study demonstrate that student preparedness is indeed critical to success. There were several important factors that determined success; the nature of approaches taken, strategies developed much earlier in the semester with a determination to perform throughout the semester. Students placed more hours of effort to complete independent tasks and attended as many tutorials and workshops as needed to gain a better understanding; in an aim to better their overall performance (10 students wanted to learn for leaning sake and were happy to continue even when they failed course). Many students devoted many more hours than allocated to the course in order to learn the material. The importance of the strategies and approaches to learning including investment of time and effort in learning environments to comprehend ideas and concepts during learning was related to success. Students who were more intense in their approaches or attempts to learn difficult or new methods and techniques performed better overall. Repeated attempts to seek clarifications during workshops and tutorials as well as outside of classes were noted in those who succeeded even when they appeared to lack confidence and beliefs about their abilities. Clearly, those who were prepared for learning in the main succeed in just that. The learning aspect is aptly noted in Claxton (1998): "Learning also comes from 'emotional intelligence', imaginative insight or after a period of rumination when the 'metaphorical apple falls on the prepared mind' (Claxton, 1998: 67); and importantly, the learner preparedness aspect is concisely summarized in "Learning to *be* is as important as learning to *do*" (Kennedy, 2002, p 437).

The repeated attempts to learn and memorize techniques or methods helped students achieve better overall results in routine as well as in the more difficult problems. Sternberg (2001a, b, c) advocated allowance for individual learning and cognitive styles in teaching to maintain student learning at efficient levels. However, Kennedy (2002) argued there were many more styles; he noted that students from

different learning styles and methods can perform equally well in other learning environments. In this study, the self-motivated and prepared learners appear to include the novel styles if they enhance their learning (attend Saturday classes or 6am meetings). It has been argued that Chinese learning is based on memory modes of learning rather than the so called deeper ways of understanding. Self-directed repeated attempts and critically intense work levels to further understand first year mathematics led to better overall results. As noted earlier, Chinese students use different styles of learning that appears to be a deeper and comprehensive type of “connective learning”; therefore, not unrelated to what is known as “deep learning”. This type of method was also noted in the work of students in this study; the students who placed high level of efforts with repeated attempts to understand the work did better overall and even scored marks higher than they believed they were capable of. The findings shows methods related to so the called ‘deep memorization’ – followed by repeated reflections over time led to success and development of confidence.

The student interview results in this study show that in contrast to Hong Kong students, many Australia and NZ students attribute their failure or success to their parents’ lower levels, their genes, and intelligences and so on; high schools students see higher performers as highly gifted students; in essence, these are often things they cannot change. However, also within this study some students identified as well prepared for learning students progressively increased their effort and determination and then applied learning strategies that led to success. Self-direction and high level of self-motivation played a part in performance of first years. Sardone (2011) noted that students often lacked self-motivation to learn and that self-efficacy played a critical role in self-motivation. Hence, some minimal level of self-motivation and determination is needed to deal with novel or non-routine tasks or else participation in learning tends to be limited.

Self-preparedness for learning can be noted in the Chinese students work even when they may not use more accepted methods of learning. It was noted that their teachers played a “wisdomic” care role outside of classroom learning (not while math content is being learned); that is, teachers identified difficulties students have and address them outside of class in a pastoral care type sessions (Tang 1996, Kember 2000; Pratt et al, 1999). In this program, tutors and the lecturer spend a number of hours with certain students during the semester both in trying to comprehend their difficulties as well as aiding them in developing a better understanding of the course content. A possible way of improving our student performance is to introduce/promote “preparedness” related factors rather early to learners; ways need to be developed that will ultimately help in “passing on” critical learning ideas as well as instilling “student preparedness” virtues for later learning environments. Pastoral care occurs in Australia but often there is barely enough time for taking care of attendance and administration related issues rather than actively engaging or imparting any deeper learning strategies. It is clear that tertiary institutions will also need to include preparedness for learning sessions within their orientation programs. Remedial sessions are routinely held but not workshops relating to student preparations for higher study (Jones and Edwards, 2010). The idea of lifelong learning is now accepted and given that students are entering universities from a diverse set of backgrounds, universities should enhance student preparedness for higher learning environments (Taylor and Mander, 2002).

The study has provided important insights into an area often ignored or not so deeply studied in the education literature. A student focus approach to research is required and students need to be made responsible for their learning rather than

teacher related factors being the focus of research into student learning. This study has highlighted the importance of student preparedness in learning environments as the key to higher learning. Clearly, student approaches to learning ought to change if they are to perform at the higher levels required of them in later learning environments. Furthering one's knowledge in an area should be a goal in all aspects of work, and therefore there is a need to teach students to approach learning environments with a persistent view to learn any work being presented. The students who fail are mostly those who lack general preparedness for learning and self-directedness and determination; in addition to general motivational strategies for learning. The failures in learning environments show and highlight the need for more research regarding the level of "student preparedness" for learning environments based on "actual classroom learning contexts"; if Australia and NZ are to be a competitive workforce in the future. The US - Department of Labor stated that "advancements in technology usually lead to expanding applications of mathematics, and more workers with knowledge of mathematics will be required in the future." It is critically important to note that both industry and government departments in all areas will need to hire mathematicians in future (US Bureau of Labor Statistics, 2011-<http://academics.nsuok.edu/stemscholars/Mathematics.aspx>), providing us with the impetus to vastly improve our number of completions in mathematics and quantitative related degree programs as well as higher degrees involving mathematics.

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Undergraduate Mathematics Around the World

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Despite the many reports highlighting the importance of mathematics in modern societies, concerns have been raised repeatedly about the need to raise the profile of mathematics education at all levels to serve the needs of research and development, to support the industry and financial sectors, and to inform policy development. This paper presents the current status of undergraduate mathematics education in the countries represented consistently in the DELTA series of conferences, and sampling five continents. Using a survey methodology, this paper summarizes their undergraduate programmes and their outcomes. It discusses what a mathematics graduate looks like, the knowledge base and set of skills they are likely to have, the different pathways into mathematics programmes and their pathways into employment, as well as the initiatives taken in their countries to support mathematics.

Keywords: undergraduate mathematics education, mathematics graduates, mathematics careers.

Introduction

The importance of the role of mathematics in modern societies to underpin research and development, to support the industry and financial sectors, and to inform policy development has been highlighted in numerous reports ([1], [2], [3]). This importance often gets underestimated by society in general, particularly in western societies, and by those in decision making positions. Many national and international reports present concerns about trends in budget allocations, quality of teacher education, and uptake of mathematics by students in schools and in higher education (see for example [4], [5], [6]).

This paper grew out of the DELTA series of conferences where the status of mathematics education has at all times been a common point of discussion and debate. (A short history of the DELTA series of conferences on the teaching and learning of undergraduate mathematics and statistics can be found in [7] and [8]). A team of conference participants undertook the task of reporting on the status of undergraduate mathematics within their own countries. They did so through a survey instrument which they completed in consultation with their colleagues, and using any official information and documents available to them. The countries chosen were: Argentina, Australia, New Zealand, South Africa, Uruguay, UK, and USA. They represent the main countries engaged in the DELTA community; they also sample five different continents.

The paper complements work done by Holton et al [9], Thomas et al [10], Varsavsky & Anaya [11] and Barton et al [12], geared towards analyzing trends in enrolments in undergraduate mathematics, not only those that lead to mathematics graduates, but also those who undertake mathematics studies as part of a technically based programme. This work focuses on undergraduate programmes and graduates of

those programmes. It discusses what a mathematics graduate looks like, the knowledge base and set of skills they are likely to have, the different pathways into mathematics programmes and their pathways into employment, as well as the initiatives taken in their countries to support mathematics.

There are many gaps in the data available to analyze and compare these aspects, and where data exist, there are many inconsistencies for the purpose of comparability. The survey approach involving academics who have operated in the higher education system for several years and who can give their inside perspective, was thought to be the most effective avenue to present a picture of the status of mathematics around the western world.

Profile of Mathematics Graduates

Despite the globalised world in which mathematics graduates are operating and will operate, it has not been possible to state what a 21st century mathematics graduate looks like. Investigation of the various undergraduate mathematics programmes confirmed that there is significant diversity in the profile of mathematics graduates of the selected countries, and even within some of the countries. This diversity is the result of the types of programmes offered, the requirements to be accepted into and complete these programmes, and the opportunities given for the development of employability skills.

Types of programmes

The mathematics graduates could have followed a generic bachelor degree (usually the bachelor of science) where students major in mathematics, or a more specialized mathematics programme. Given the flexible nature of a generic bachelor degree, the mathematics studies taken within such programmes could comprise anywhere between one third and 90% of the studies for the degree. The different models currently in place are outlined below:

- *Argentina:* Mathematics is offered through a five year specialized degree, called licenciatura, with more than 90% of studies involving mathematics and including a significant research project in the last year. The licenciatura is the normal pathway to studies for a doctoral degree.
- *Australia:* In Australia there has been a shift over the last years away from the specialist bachelor degrees towards generic three year bachelor of science degrees. Very few universities still offer a specialized mathematics degree. In the generic degree, a student can take as little as one third of their studies in mathematics or as much as 90%. The best students can choose to add on an additional intensive year focusing on further mathematics studies and a research project to graduate with a bachelor degree with honours. The honours degree is still considered an undergraduate degree and is the normal pathway to doctoral degree studies.
- *New Zealand:* Offers both a generic bachelor of science or a specialized bachelor of science with between 50% and 65% dedicated to mathematics studies. An additional honours year is available.
- *South Africa* is also in the process of shifting from specialist degrees to a generic bachelor of science where students can combine mathematics studies with studies in other disciplines. The South African model will require half of the bachelor studies to be in mathematics. An additional honours year is available.

- *Uruguay*: Like Argentina, offers a licenciatura, which is a four year specialized degree, with more than 90% of studies involving mathematics. Special programs are offered for the qualification of teachers in the tertiary non-university sector.
- *UK* offers the three year bachelor of science, with one or two specializations, which means that a graduate in mathematics could have done 50% or 100% of studies in mathematics. The MMath is a four year research track degree, and many institutions also offer a four year bachelor of science which includes a sandwich year for an industry placement. Foundation degrees, which build work-based learning with specific employers into a programme, are also available, but have not proven popular in Mathematics, with only one course being offered nationally. Graduates from any of the full degree programmes can progress to a higher degree (MSc or PhD) although a greater proportion of MMath students are likely to choose to do so.
- *USA*: In the United States students have traditionally undertaken mathematics studies as a major within a bachelor science or a bachelor of arts of four years duration, with between one third and one quarter with studies in the major area of specialization. Students planning to go on to graduate study in mathematics get the same degree as other mathematics majors, however, they may elect to take a few more courses, or more advanced courses in mathematics

Comparability of standards

The curricula for these programmes are set by the individual institutions that offer them, and in most cases there are no mechanisms for comparing standards amongst the different institutions within their countries. The only exception is the UK, where the European approach was followed to define a nationally agreed benchmark statement for mathematics [13]. Australia is now also in the process of developing discipline standards. In both cases, in Australia and the UK, the benchmark statements attempt to provide a high level definition for the minimum or threshold knowledge, understanding and skills (both discipline related and generic skills) of a mathematics graduate. In the UK courses must demonstrate that their learning outcomes meet the requirements of the national benchmark statement in order to gain validation from the host university, and according to current developments, it is likely that this will also be the case in Australia in the near future. However, although the defined threshold skills are used as guidelines for teaching and assessment, there are no current mechanisms in place to ensure that every mathematics graduate has in effect attained the indicated minimum standards.

Employability skills

The skills mathematics graduates are equipped with to be able to provide a meaningful contribution towards supporting industry and driving innovation, research and development of our world are crucial. In addition to a sound knowledge base and problem solving skills, graduates must have developed skills that would allow them to operate effectively in the workplace, which has increasingly become globalised, multi-disciplinary and team based. The so called employability or generic skills are just as important as the academic skills. Yet, there are no clear indications that all mathematics graduates possess such skills when leaving university. Industry placements or internships where student are immersed for some time in a workplace environment are rare. The exception are some UK institutions, particularly the newer

universities, which provide work placement opportunities to their students giving them the option to take a “sandwich” fourth year. In the UK, the importance of these skills was recently explicitly recognized by the funding council, who has recently required each University to publish an Employability Statement, setting out that institution's approach to developing graduate employability [14]. The importance of work experience has also been recognized by employers, with reports that “a third of graduate jobs will go to people with work experience” [15]. Also, as a result of the Burgess Report [16], which explored ways of providing a better measure of student achievement than the degree classification, all universities in the UK will be required, by 2011/12, to provide graduates with a Higher Education Achievement Report (HEAR) [17]. In addition to data regarding academic grades the HEAR will provide details of all accredited achievement, in particular key or generic employability skills, so that employers have a much better picture of each graduate's capabilities.

The UK and Australia are the only countries where data is collected about graduates' perceptions of their readiness for work. In both countries, employment data are collected from students six months after graduation, with questions on students' own perceptions of having developed so called “generic or employability skills” [18 & 19]. In the UK, 77% of the mathematics graduates surveyed in 2010 agreed with the statement “As a result of the course, I feel confident in tackling unfamiliar problems”, placing mathematics in the 28th position out of 42 subjects. For the survey statements “My communication skills have improved” and “The course has helped me present myself with confidence” mathematics was ranked last. This has consistently been the case for the last three years (for full details, see <https://maths.shu.ac.uk/NSS/Skills2.php> which shows, perhaps paradoxically, that mathematics is one of the subjects with which students are most satisfied overall, coming, for instance, in second place overall in 2010. One inference that could be drawn from this is that students may not consider the development of personal skills to be very important, which is a problem in itself!).

In summary, graduates of mathematics undergraduate programmes form a mixed bunch. Their mathematics knowledge foundation and problem solving skills might have been built on as little as the equivalent to one year of full time studies, or as much as four years. Furthermore, their skills base to operate in a workplace environment may differ widely, although it is more likely than not that they have not had any formal opportunities to develop such skills.

Access to Undergraduate Mathematics

Entry requirements

Completion of mathematics studies in senior secondary school, with a certain level of achievement, is the normal entry requirement into undergraduate mathematics programmes. However, in many countries mathematics is not compulsory in senior secondary school; students have the choice whether to study mathematics or not and to what level. Of the countries chosen for this study, mathematics is a compulsory study in senior secondary school only in South Africa and in the South American countries. In South Africa students have the choice between the traditional mathematics curriculum or the skills based study Mathematics Literacy, with only the former providing entrance to university studies in mathematics. In South America the level of mathematics depends on the overall specialization students take in secondary school (technical, business oriented, science).

The rather alarming reality is that a large proportion of secondary school students

choose not to take mathematics in their senior level and hence close their doors not only to careers in mathematics but also to careers in science, technology, engineering, and in several countries also to careers in business. In the USA, only half of high school graduates take mathematics during their senior year [20] with just 18 percent taking the advanced level (Algebra II) [21]. This situation is similar to South Africa where equal numbers of students take the mainstream Mathematics and Mathematics Literacy studies. For New Zealand and Uruguay, it is estimated that this number is between 30 and 40%. In Australia, about 80% of the students study mathematics to age 18; 50% take it only to an elementary level, 20% to an intermediate level, and 10% to an advanced level [22]. These figures are comparable to the UK, where 100% of students study mathematics to age 16, but only 9% of the students study mathematics to an advanced level (A2 level). An increasingly popular option in the UK and in Australia is the International Baccalaureate, in which all students study mathematics to age 18 as part of the Diploma programme [23]. In the USA the uptake of the International Baccalaureate remains small.

In the USA, the Advanced Placement Calculus program—college level course in calculus offered in high schools—is becoming increasingly popular [24]. The number of students who take the AP Calculus exam is around 50% of the high school students who enroll in a course entitled “calculus,” putting that number at around 600,000, or one-third of the traditional college-bound high school graduates each year. Despite the greater numbers of students arriving with advanced placement, there has been no evidence of growth in the number of students continuing on to several variable calculus, linear algebra, differential equations, or advanced mathematics. Much of this increase in the number of students taking calculus in high schools is driven by college admissions at selective colleges and universities and by high schools encouraging students, to do so—whether or not they are adequately prepared. Only between a quarter and a third of those who study calculus in high school, receive and take college credit for their calculus course.

Pathways

Given this low uptake of mathematics in secondary school, the higher education sectors of all but the South American countries have sought to address the shortage of graduates with mathematical skills, and made efforts to provide bridging studies as a pathway into studying mathematics at an undergraduate level, which in most cases results in students extending their studies. Students are usually selectively admitted and higher education institutions certainly provide an otherwise denied opportunity for such students to succeed.

- In the UK, students can follow an access course at a local college, a foundation year or preparatory year at university, or they can sign up with the Open University that has no prerequisite requirements. A recent study showed approximately 1770 students studying a foundation year in Mathematics, the vast majority (over 90%) with the Open University. No national figures exist on related student success rates, but for the Open University around 30% progress to an Honours Degree. There is a wide range of outcomes for students progressing from a preparatory year, some finding the going very difficult with others becoming the highest achievers in their cohort [25].
- In New Zealand, there are some pre-entry level mathematics courses as well as the Tertiary Foundation Certificate, which can prepare students for the required mathematics background. This certificate comprises a selection of subjects, for university programme applicants who lack the grades to gain

admission or want to return to study; students combine English and Mathematics with two subjects from Biology, Chemistry, Geography, History and Physics, Polytechnic institutions also offer other foundation courses or diplomas that could lead onto mathematics degrees. In South Africa, most universities offer pathways for students who have not met the mathematics university entrance requirements but show potential. A typical programme for such students is the four year extended bachelor of science, admission to which is dependent on the results of the institutional proficiency test. The programme attracts more applicants than positions available. A fair number of students use the first year of this programme as a stepping stone into other fields such as engineering and medicine. This is a new approach and hence it is too early to determine the effectiveness of this pathway opportunity.

- In Australia, most universities offer pathways within the bachelor of science programme for students who have not completed the appropriate mathematics studies at senior secondary school, but very few choose to take this path and with mixed success [26].

National initiatives

In addition to pathway programmes developed by individual institutions, survey participants were asked to report on any recent national initiatives or projects to encourage more people to study mathematics at university. With the exception of the South American countries, there have been varied levels of activity around the world to raise awareness of the need of people with skills in mathematics, and more generally in science, and to support programmes aiming to attract more students to mathematics, both at secondary and tertiary levels.

The UK has clearly taken the lead in recent years to address the mathematics graduate shortage problem. The reforms introduced to the 16-18 curriculum in the UK through *Curriculum 2000* led to a substantial reduction in the number of students choosing to study mathematics at university. The reforms were referred to as "an utter and complete disaster" and action was required to correct the problem [27]. These changes had a significant impact, and from a low point around the period 2002-2004 student numbers have gradually increased. The projects with greatest impact include:

- The *More Maths Grads project* (2007-10) [28] was set up to "develop, trial and evaluate means of increasing the number of students studying mathematics and encouraging participation from groups of learners who have not traditionally been well represented in Higher Education". The project has been a remarkable success, and mathematics is now the second most popular choice for study in senior secondary school, second only to English. Further mathematics, a qualification taken by the most able mathematics students that both broadens and deepens the A2 syllabus [29], was named as the fastest growing A-level subject in the country in 2010 as entries soared by 11.5 per cent in just 12 months [30]. The Higher Education Funding Council for England published a note on 11 March 2010 into the strategically important and vulnerable subjects stating that mathematics numbers had risen by 6.8% since 2005 [31].
- The *More Maths Grads project* has been followed by the new National *HE STEM (Science, Technology, Engineering and Maths) Programme* [32], with the aim to encourage the exploration of new approaches to recruiting students and delivering programmes of study within the STEM disciplines.

- A new project, funded by the National Higher Education STEM programme—*Developing Graduate skills in HE Mathematics Programmes through Case Studies of Successful Practice*—is aiming to begin to address the need to produce graduates better prepared for the workplace by identifying successful curriculum-based practices for embedding employability skill development. These case studies are intended to be short, and have been catalogued according to the specific skills each initiative is aiming to develop. The intention is to produce a printed booklet. A printed version of the booklet will be available in 2011, downloadable from <http://maths.shu.ac.uk/msor/graduateskills/>.)

However, due to recent government cuts to funding for UK Higher Education, it is difficult to see that this trend of improvement will continue. A major factor in student recruitment for 2012 and beyond is the planned approximately three-fold increase in student tuition fees, combined with a near total removal of government subsidy for universities. At the moment, the government is signaling that only STEM subjects will be subsidized, but the level of the subsidy, if it happens, has not been made clear.

In Australia there has also been some recent vigorous activity. The shortage of mathematics skills has been raised repeatedly over the last 10 years after the review of Mathematical Sciences in Australia [6]. The government, who has main responsibility for higher education was slow in responding, but finally in 2008, it decided to act to address the findings of the review and of several other reports pointing to the need to support mathematics, and in particular to increase the numbers of competent mathematics teachers. The main policy put in place was to support access to mathematics studies by reducing the higher education fee students pay for mathematics and science studies to almost a half. Observation of recent enrolment data indicate that there has been an increase in science applicants in the last two years, but it is not yet clear whether this is due to the lower fees or other factors such as higher unemployment rates due the global financial crisis.

The American National Science Foundation has for several years run the *Science, Technology, Engineering, and Mathematics Talent Expansion Program*, whose purpose is “to increase the number of students (U.S. citizens or permanent residents) receiving associate or baccalaureate degrees in established or emerging fields within science, technology, engineering, and mathematics (STEM)” [33]. The need for such programmes is the topic of much discussion in the United States, but it is not clear how effective and widespread they are. According to the Conference Board of the Mathematical Sciences 2005 study [34],

The total enrolment growth in four-year colleges and universities during the 1995-2005 decade outstripped mathematics and statistics enrolment growth, and in the fall 2005 there were many more American college students taking substantially less mathematics and statistics courses than did their predecessors a decade earlier. Four-year colleges and universities saw fall term enrolments in mathematics and statistics rise by about 8% between 1995 and 2005, at the same time that total enrolment in four-year colleges and universities grew by about 21%. The problem was even more pronounced in the decade’s last five years, between fall 2000 and fall 2005, when mathematics and statistics enrolments in four-year colleges and universities actually declined, at the same time that total enrolment in four-year colleges and universities rose by about 13%.

In New Zealand, only local initiatives have been reported. For example, the Auckland Community of Undergraduate Learning in Mathematical Science has developed a newsletter, a lecture series, and the project A Vision of Senior Secondary

and Undergraduate Mathematical Sciences (SSUMS) [35].

No formal national initiative to encourage more people to study mathematics at university has yet been introduced in South Africa. However, the South African Mathematics Foundation has launched initiatives to create an awareness of careers in the mathematics field through initiatives such as distributing booklets on Careers in Mathematics and by establishing a national Mathematics Week to promote mathematics, for example. Despite these efforts, enrolment figures for degrees in pure mathematics have not shown an increase yet. There is, however, an increasingly large demand for the programmes in Actuarial Science and Financial Mathematics despite their stricter entrance requirements.

Finally, in contrast with the examples provided above, it would appear that the mathematics issue has not yet made it to the national agendas in New Zealand and the South American countries.

Funding for students

Only in South America is there free access to study mathematics, and in general to higher education. Fees are to be paid in all other surveyed countries. In most cases student access a loan, and there are scholarships for under-represented groups. Australia is the only country where there is now a fee reduction for those students who study mathematics or any other fundamental sciences. As indicated above, it would appear that this is an effective measure to attract students to mathematics, but it may be too early to draw conclusions.

Outcomes and Career Paths

Graduate statistics

Figures on the number of students studying mathematics at university level, and the number of mathematics graduates are difficult to obtain. This has improved over the last years; however, the approaches taken in reporting these figures both internally and through the OECD, differ greatly. To give just one example, in Australia the reporting of bachelor of science graduates of individual institutions to government was for some time inconsistent, with some reporting graduates with mathematics majors under the broader specialization of “physical sciences”. Furthermore, even if data were available for each country, the different programmes under which mathematics can be studied still makes the comparisons rather difficult, and hence only estimates can be provided.

Our sample includes countries with averages of mathematics graduates as a percentage of the total number of graduates both above and below the 1% OECD average for mathematics graduates [1], but as pointed above, there are inconsistent methodologies in the gathering of data used to arrive to this figure.

In Australia, the statistics arrived at by Dobson [36] from the figures the individual institutions reported to government, indicated that around 2005, almost 7% of the students were undertaking studies in natural and physical sciences (as a major study within a bachelor of science, or as a compulsory study towards other degrees such as engineering). Of these, just above 13% were studying mathematics, which is just more than 1% of the total student population in higher education. A similar analysis indicates that mathematics graduates count for about 0.4% of all graduates in any given year.

In the USA, this number is higher; inference made from government statistics

indicate that the proportion of mathematics graduates in 2008 was around 1.5% ([34] and [37]). This seems to be similar to the UK, where the proportion of students studying mathematics is about 1.5%.

Retention has been a major focus of many UK higher education institutions, particularly the 'new' post-92 universities. Typical retention rates remain around 80-85%, but there will be considerable variation from institution to institution.

In South Africa, about 0.1% of school leavers enroll in a programme leading to a mathematics degree; if actuarial and financial mathematics is counted, then this figure rises to 0.4%. In Uruguay, statistics on mathematics graduates include also mathematics teachers who graduate through the Institute of Teaching rather than a university; both groups account for less than 5% of all graduates. Argentina is the South American country with the highest participation in higher education; however, mathematics graduates account for only 0.3% of all graduates [11].

What do mathematics graduates do?

If determining the number of graduates in mathematics is hard, knowing and comparing what they do after they graduate seems to be even harder. There is some systematic gathering of information about graduate destinations in general, not just specifically mathematics graduates.

The UK and Australia, and to some extent New Zealand collect data about their graduates on a regular basis ([37], [38], [39], [19]). The data is collected through a survey administered to graduates about four to six months after graduation. The survey asks for current employment status (available/not available for work, or employed), the industry sector of those employed, and whether they are undertaking further studies.

These data sources indicate some commonalities across the countries that use graduate destination surveys. Within the first year after graduation, typically 30-40% of mathematics graduates are engaged in full time study. In the UK, approximately 15% of graduates each year take a Post Graduate Certification in Education (PGCE) and enter the teaching profession. It is also known that in countries with the generic bachelor of science the further studies graduates take are more likely to be at honours level. Some of the graduates continue with PhD or masters studies within mathematics or other complementary discipline. There are also indications that for those who are working, the starting salaries (where known) are also typically slightly higher than the average for all graduates.

Graduates in all countries find employment in business and finance, education, government, health and private industry. However, there are differences regarding the sector of these employment destinations. Teaching (particularly in the higher education sector) seems to be a more common destination in Australia (15%) than in the UK (only 5%), and the business and finance sector employs 50% and 30% of the UK and Australian graduates respectively.

In contrast, the USA, South Africa and the South American countries have not yet introduced this practice and hence the information about graduate destinations is simply anecdotal or common knowledge within the academic sector. In Uruguay, for example, it is known that most mathematics graduates are employed in the education sector, and that their starting salary is a third of that of an engineer.

Conclusion

This paper presents an overview of the current standing of undergraduate

mathematics in a sample of countries represented in the DELTA community that also represents five different continents. There are stark differences in the approaches taken by these countries, from the mathematical skills base developed in secondary school, through to the programmes offered to prepare mathematics graduates, and to following up on graduate outcomes.

This comparison leads to the expected but nonetheless very important conclusion that the countries with the larger investment and creativity in programmes to attract more students to do mathematics at secondary school level and in higher education are producing more mathematics graduates. Prospective students need to be shown the value of mathematics in modern society, and be made aware that mathematics skills open a large variety of job opportunities through which they could contribute towards solving problems the human race is facing. Higher education institutions must ensure that they adequately prepare graduates to be able to make that contribution. Although a strong mathematics skill base is essential, a greater emphasis needs to be placed on developing employability skills that will enable graduates to adapt to different working environments. Following up on graduate destinations is important not only to measure the institutional outcomes, but also to strengthen the message that mathematics graduates do find employment, that they contribute towards solving problems across all industry sectors, and that their salaries are comparable to, if not better than other professions.

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SECTION C

ABSTRACTS OF PRESENTED PAPERS

Neither Flesh, Fish nor Good Red Herring: Mathematics for Science Students

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The Australian education system is typically characterized by strong disciplinary divides between mathematics and science, commencing at primary school and often reinforced at the secondary and tertiary levels. However, modern science, including biological science, is increasingly quantitative and mathematical in nature. This anomaly has been identified internationally, but institutions appear to be struggling with how to resolve it. In this talk we will describe an approach introduced at the University of Queensland and which has been refined over four years. The talk will focus on what worked well, what failed miserably, how students and colleagues responded, and what we would do differently next time.

A Suggested Framework of a Remedial Teaching Strategy in Correcting Common Mistakes in the Processes on Fractions and Decimals for Preparatory Year Program Students at Taibah University

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One of the most important recent trends in the teaching of mathematics has confirmed the results of educational research on the existence of individual differences between students in their academic achievements. Accordingly each teacher needs to take this into consideration when planning for teaching; to know the levels of students and the mistakes that they usually make, and to develop the necessary solutions, and provide effective therapeutic material to them. The current study aimed at designing a framework of a remedial strategy for correcting common mistakes committed by the preparatory year students in the operations on fractions and decimals. This goal emerged from the problem of a recent questionnaire, which identified the existence of some common mistakes in the preparatory year program. The need for this remedial strategy comes from the importance of fractions and decimals in the future technical life of students. Even though computers and calculators solved most calculation problems, but mental calculations and the concept of operations on fractions are still needed for creativity and necessary estimations. To reach this goal applying of different tools were necessary for the study such as tests and questionnaires. The study found a number of key results such as: (17) errors in the combining of fractions, which reached the degree of common errors identified in this study as (25%) or more, of them (6) common mistakes in the skill of adding fractions, and (7) common mistakes in the skill of the process of adding fractional numbers, and (4) common mistakes in the skill of adding decimal places, (16) errors in the subtraction of fractions which amounted to the degree of common error identified in this study (25%) or more, of them (5) common mistakes made in the skill of subtracting fractions, and (7) common mistakes in performing the skill of subtraction of fractions, and (4) common mistakes in the skill of the process of subtracting decimals (9) common mistakes in multiplying fractions reached the degree of common errors identified in this study (25%) or more. (4) common mistakes were identified in the skill of multiplication of fractions, (5) common mistakes in the skill of multiplying fractional numbers. (27) errors in the division by fractions, which reached the degree of common error were identified in this study (25%) or more, of them (15) common mistakes in the skill of dividing fractions, and (12) common mistakes in the skill of processing fractional numbers.

In light of the results of the study, the researcher suggested a framework of a remedial teaching strategy in correcting common mistakes in the processes on fractions and decimals for preparatory year program students at Taibah University.

Teachers' Oral Communication Behaviours in the Mathematical Problem Solving Classroom: The Ghanaian Students' Perspective

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The discourse, which occurs between teachers and students in the classroom, contributes immensely to students' mathematical understanding. The usefulness of oral communication as a discourse process to improve students' mathematical understanding has been highlighted in numerous studies including Cobb (1994). This study investigates the types of oral communication behaviours, which Ghanaian students perceive as the oral communication behaviours mathematics teachers use during problem solving, and, why mathematics teachers use such behaviours in the classroom. This study further investigates students' reactions to teachers' affinity to using those types of behaviours. The participants in this study comprise 180 undergraduate mathematics education students at a university in Ghana. The research design for this qualitative study was grounded in a phenomenological framework, a discovery-oriented method where researchers often possess an attitude of openness in allowing the unexpected meanings of a phenomenon to emerge (Giorgi, 1997). The results of this study indicate that, teachers rely on effective oral communication behaviours, which includes speaking loudly and clearly, and, repeating methods until all students understand the procedures. Students note that, teachers do everything under their purview to make mathematical problem solving, enjoyable and interesting. The students further note that, improving teachers' instructional delivery in schools through effective oral communication behaviours, is one of the main preoccupations for teachers. This study concludes that, oral communication behaviours of teachers could assist researchers and policy makers to identify key mathematics content areas that lack the needed instructional attention, and enable them to design appropriate instructional methods capable of improving students' conceptual understanding (Cohen & Hill, 1998; Fennema, Carpenter, Franke, Jacobs, & Empson, 1996). Further, the study notes that oral communication behaviours of teachers could enable researchers and policy makers to identify teachers' instructional lapses. This could guide researchers and policy makers to effectively design appropriate instructional methods to be implemented in schools. Additionally, this study provides an insight into the instructional challenges facing mathematics teachers during problem solving. With knowledge of teachers' oral communication behaviours in schools, this study could open the window for mathematics educators to focus on the myriad of problems facing the teaching and learning of mathematics.

Quantitative Analysis of a Moore Method Course in the U.K.

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The Moore Method is an enquiry-based learning pedagogy devised by the Texan topologist R. L. Moore in the first half of the 20th Century. The Method centres on learning by problem solving, with students working individually to solve problems before presenting their solutions to the class. The group then agrees whether the solution is correct or if there are improvements to be made. The lecturer acts as a facilitator, posing the problems that the students are to answer, and asking questions during the students' presentations. No direct instruction takes place.

At Birmingham we set up a first year Moore Method course in the academic year 2004/05 to develop students' problem solving skills. I will present an analysis of the progression of the 60 students who have taken the course since 2005/06, comparing them with the results of over a thousand other students in the department. The data used encompass 10,000 exam results over six years, and show that students who have taken the course perform better than their peers in a range of other courses in the School of Mathematics.

Besides the extensive quantitative gathered, we have also conducted interviews with a number of students who had taken the course in previous years. A majority of these students report that the course aided them with their approach to study in other courses, through to their final year of their degree.

The Challenges of Upgrading Practicing Mathematics Teachers in South Africa

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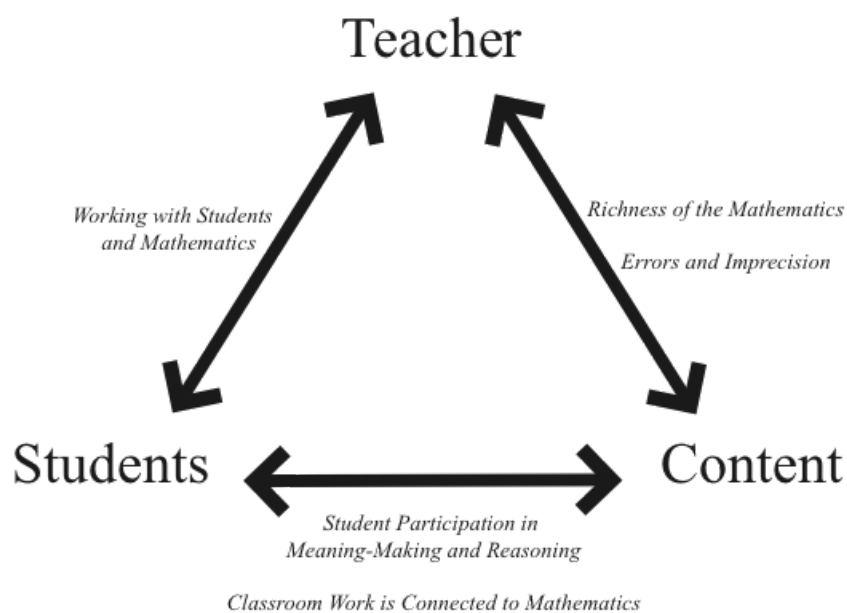
A particular inheritance of the apartheid system in South Africa was the disparate teacher training offered at different colleges of education. There was much variation in the teacher preparation amongst the different races because most training was carried out in racially segregated colleges of education which were academically isolated and ineffective. Many mathematics teachers have a three year college teaching diploma, with little post-secondary mathematics. Most (not all) of these colleges were seen to as to be producing teachers of poor quality, resulting in widespread education problems. This led the government to shut down all colleges of education in the 1990's. This closure has resulted in a persistent problem of large numbers of under-qualified practicing teachers in the system. The numerous curriculum reform processes in the South African education system have added to this challenge by requiring teachers to be retrained to teach the new curriculum. For example, concern about the low levels of numeracy in our adult population has translated into the introduction of the subject Mathematical Literacy (ML) as a fundamental subject in the Grades 10-12 band in order to help develop numeracy skills. Its purpose is not for learners to do more mathematics, but more application and to use mathematics to make sense of the world.

Although well intentioned, the decision to introduce of ML in all secondary schools resulted in a challenging situation, considering the fact that there was already large numbers of under-qualified practicing teachers in the system. There were few mathematics teachers available, who understood the subject and who were willing to facilitate the subject with the pupils. Thus the teaching of the subject ML is mainly done by out-of-field teachers. Many universities have offered part-time programmes designed for the dual purpose of upgrading under-qualified teachers as well as retraining out-of-field teachers to teach ML. In this paper by considering one such programme, we examine the extent to which these dual aims have been met. The study was conducted with 588 teachers, 289 of who were considered as upgrading while 290 were deemed to be retraining. Data was collected from student records, examination records of performance in different modules as well as interviews with a selection of each group. By looking at the success rates of these two groups of teachers (upgrading and retraining) we explore whether the programme has been successful in straddling between the two aims. This paper has implications for teacher training globally since using out-of-field teachers to teach mathematics and science is widespread, and the implications of retraining such teachers is of interest to all.

Exploring Mathematical Quality of Instruction

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When we observe a mathematics lesson, what we think about that lesson and how we describe the important features of it can differ greatly. For example, one particular lesson has elicited responses such as “This was really strong mathematics instruction” and “The teacher’s use of the mathematics was all wrong.” With such varied reactions to the same lesson, what common language or framework might we use to think about and discuss mathematics instruction?



Through watching hundreds of mathematics lessons, by looking at attributes common to lessons at different grade levels and content areas, and from a review of the mathematics education research, the Mathematical Quality of Instruction instrument (MQI) was developed to reliably capture and describe the work that teachers do with students around mathematical content. The MQI is a lens through which to observe the nature of the mathematical content available to students during instruction, as expressed in teacher-student, teacher-content, and student-content interactions (Cohen, Raudenbush & Ball, 2003). The MQI is based on the perspective that the mathematical work that occurs in classrooms is distinct from classroom climate, pedagogical style, or the deployment of generic instructional strategies. As such, it provides a balanced view of the numerous elements that comprise a mathematics lesson.

The MQI looks at five dimensions of mathematics instruction that are found within the interactions seen in the figure above: (1) *Richness of the Mathematics* includes attention to the meaning of mathematical facts and procedures (e.g., explanations of mathematical ideas and drawing connections among different mathematical ideas) and engagement with mathematical practices and language (e.g., multiple solution methods or mathematical generalizations); (2) *Working with Students and Mathematics* looks at whether teachers

hear and understand what students are saying mathematically and respond appropriately; (3) *Errors and Imprecision* refers to mathematical errors and possible confusion of the content by the teacher; (4) *Student Participation in Meaning-Making and Reasoning* captures the ways in which students engage with mathematical content. Specifically, do students ask questions and reason about mathematics, do they provide mathematical explanations spontaneously or upon request by the teacher, and what are the cognitive requirements of a specific task?; (5) *Classroom Work is Connected to Mathematics* considers whether classroom work has a mathematical point or whether portions of instructional time are spent on activities that do not develop mathematical ideas (e.g., administrative issues or classroom management).

In this interactive session, the full MQI instrument will be introduced and its language, focus, and rationales will be presented. We will look at specific examples of secondary mathematics lessons using the MQI and we will consider what such a lens affords conversations about particular lessons. We will discuss what the MQI offers to teachers and possible uses of the MQI, such as lesson planning, lesson study groups that focus on content areas, and teacher preparation.

A Story of Teaching Mathematics to Economical Sciences Students

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Large numbers of students enrol at the department of economics at our university. The first year mathematics curriculum for these students include topics such as equations, financial mathematics, functions and graphs, systems of equations, matrices, linear programming, calculus and integration. This module historically had a lower than normal pass rate, and was in some cases an impediment to students in obtaining their degree. This posed a problem. Because mathematics is a service subject for the economical sciences we realised the importance of networking with lecturers in the economics department. They reported that the students are not mathematically competent and that they have difficulty with the application of mathematics in their economical subjects.

One of the myths in the teaching of mathematics at university level is the existence of a context-free universal content [1]. Previously the content of first year mathematics modules tended to be generic and consisted of basic knowledge and skills learned in a mathematical context devoid of applications to real-life situations. It was taken for granted that there are some core elements of mathematics that needs to be learned by all students before applications can be made. However results show that these approaches suppress students' interest and hampers interdisciplinary applications [1]. Most students are unable to connect their mathematical knowledge to other disciplines without support. Therefore teaching mathematics using the context of the applicable disciplines may enhance students' potential to apply their mathematical knowledge

We adapted the mathematics content of the module for the 2011 intake by implementing a realistic economic context as a setting for mathematical problems. For example, instead of using $y = mx + c$ to represent a straight line, we used the form $Q = aP + b$ which is used in economics. We called these functions demand/supply curves, as used in economics, instead of straight lines. When dealing with derivatives we applied this in the economics domain as marginal cost and marginal revenue. In order to achieve this we prescribed a textbook that integrates mathematical concepts and economic principles. The assessment was modified to mirror the economic context used.

A comparison between the results of the module in 2010 and 2011 showed that the group of 2011 outperformed the group of 2010. According to the statistics there was a much better pass rate in 2011(82,6%) than in 2010(70,9%). Interviews were held with some of the students as well as some of the economics lecturers to determine whether the new way of introducing the mathematics improved the students' understanding of the mathematics in the economics module. The results will be discussed in the final paper.

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The Fractal Structure for Generic-Competency Development Based on a Mathematics Course Experience

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In a society characterized by the fast generation of knowledge coming from all fields, as well as by the speed at which such knowledge loses validity, it is essential to combine globalization, transdisciplinarity and educational processes based on competency development, rather than focusing on knowledge acquisition. This educational approach requires the integration of knowledge, boundary-fading actions and the breaking of established dichotomies and gaps among academic, socio-cultural and employment frameworks.

In this sense, incorporating transversality into the field of education is necessarily associated to taking a stand, to committing with conditions and strategies to deal with diversity and with the integral education of individuals within a global and changing environment through competency development, providing students with the basis and the necessary skills to continue learning.

More specifically, the identification of generic competencies as benchmark for different contexts implies an opportunity. The above mentioned competencies provide a vertebral line which promotes coherence among didactic proposals at all educational levels. This action requires knowing and understanding their concept and meaning, appreciating their value and the need to integrate them into academic curricula.

One of the first consequences derived from these renovations in the educational field is that major changes are required from every system stakeholder, including the conception of teaching proposals and institutional approaches aimed at providing teaching training, in order to make significant changes in the implicit dynamics of the teaching-learning process. As long as these suggested transformations are achieved, instructional designs aimed at increasing the student's independence in their own learning process will be granted.

This research presents the experience of a curriculum reform carried out in a Venezuelan higher education institution, *Universidad Metropolitana*, based on the above mentioned approach. Once the conceptual foundations of this reform - whose main pillars are lifelong and autonomous learning, well-grounded general and basic education, and competency-based learning were defined – a start-up process having the self-similarity characteristic of the fractal structure was established as the implementation model.

This implementation model can be organized in three major stages: the incorporation of transverse axes in each subject as a first attempt; the subsequent definition of generic competencies for all graduates from the various careers organized in transverse axes and, finally, the development of these large transverse axes for each subject. In other words, this three-stage scheme represents the core model to be replicated in each of the subjects that make up the curriculum. Particularly, this research accounts for the experience of one subject in the field of mathematics with new students entering the university.

La Estructura Fractal Para el Desarrollo de Competencias Genéricas Desde la Experiencia en un Curso de Matemática

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En una sociedad caracterizada por la acelerada generación de conocimientos en todas las áreas del saber, así como, por la velocidad a la que estos pierden vigencia, se hace indispensable conjugar globalización, transdisciplinariedad y procesos educativos basados en desarrollo de competencias, en lugar de aquellos cuyo foco está en la adquisición de conocimientos. La formación así concebida exige integrar saberes, desdibujar fronteras y romper las establecidas dicotomías y brechas entre el marco académico, el marco sociocultural y el marco laboral.

En este orden de ideas, la incorporación de la transversalidad en el ámbito de la educación está asociado, necesariamente, al compromiso con una postura, condición y estrategia para enfrentar la diversidad y la formación integral del individuo en un entorno global y cambiante, mediante el desarrollo de competencias que brinden a los estudiantes las bases y la preparación necesaria para seguir aprendiendo.

Más específicamente, la identificación de competencias genéricas como elementos de referencia para distintos contextos representa una oportunidad. Las mencionadas competencias proporcionan una línea vertebradora que promueve la coherencia de las propuestas didácticas en todos los niveles educativos. Ello exige conocer y comprender el concepto y sentido de las mismas, apreciar su valor y la necesidad de integrarlas en los programas de formación académica.

Una de las primeras consecuencias que se desprende de estas renovaciones en el ámbito educativo es que se solicita a todos los actores del sistema, cambios importantes desde la concepción misma de las propuestas de formación, pasando por la necesidad de respuestas institucionales que apunten a la capacitación del profesorado, para concretar cambios significativos en la dinámica implícita del proceso de enseñanza aprendizaje. En la medida que se logren las transformaciones sugeridas se lograrán diseños de instrucción orientados a que cada vez el estudiante obtenga más independencia en su aprendizaje.

En esta investigación se presenta la experiencia de una reforma curricular en la dirección de lo anteriormente expuesto, en una institución de educación superior venezolana, la Universidad Metropolitana. Una vez definidas las bases conceptuales de esta reforma cuyos pilares fundamentales están en la educación permanente, el aprendizaje autónomo, la sólida formación general y básica y la formación basada en competencias; se inicia un proceso de puesta en marcha que toma como base para establecer su modelo de ejecución la característica de autosimilitud de la estructura fractal.

El modelo de ejecución de la reforma puede ser organizado en tres importantes etapas: incorporación de ejes transversales en cada asignatura a modo de ensayo; la posterior definición de competencias genéricas para todos los egresados de las distintas carreras organizadas en ejes transversales y, finalmente, el desarrollo de esos grandes ejes transversales en cada asignatura. Es decir, este esquema de tres etapas representa el núcleo a replicar en cada una de las asignaturas que conforman los planes de estudio. En particular, en estas líneas se recoge la experiencia descrita desde una de las asignaturas del área de matemática para estudiantes de nuevo ingreso a la Universidad.

A Statistical Consulting Capstone Unit: What to Assess?

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Statistics education is becoming more important as technological changes make the availability of large amounts of data part of our daily lives. However, young Australians' wishes to become statisticians are not increased with the increased need of the nation. As statistics educators, we are always on the lookout for new recruits through our service units so that we can help our nation educate an appropriate number of statisticians. An important question, then, is what should go into a statistics major's curriculum?

With the curriculum change that took place at Macquarie University, Australia in 2008, we were able to include a Capstone Unit in our Statistics Major, in which we aimed to introduce our students to the skills of statistical consulting. The main aim of this unit is to prepare our graduates for the workforce by providing them opportunities to carry out authentic statistical consulting before they graduate. The content of the unit is more concerned with so-called "soft" skills (i.e. oral communication, report writing, asking the right questions, and learning from peers) than technical concepts (i.e. more statistical techniques). This brings us as educators to uncomfortable territory - the assessment of these skills. We have learned through our own education and up to this point through our teaching, in how to assess application of statistical techniques and what to be aware of technically when educating our students. However, we commonly focus less on the soft skills, especially in undergraduate units, since we feel we don't have enough room in our curriculum to accommodate such skills. Also, many of us wonder if we in fact can or should teach report writing, communication or such soft skills to our students in statistics, as after all we are not necessarily experts in these areas.

In these uncharted waters, we designed our unit to cover the following topics: a) the human side of statistical consulting; b) asking the right questions of clients so that we can translate their problem into a statistical problem; c) learning to work in a group - after all, no statistician works in isolation except perhaps theoretical statisticians; d) writing a statistical report that makes sense to a client, and more. It is neither easy nor desirable to come up with formal lecture plans for these topics, especially when we have unsolved real problems and unknown real clients to deal with. However, we have managed by being flexible and being willing to prepare some of our lectures on the basis of student requests and requirements. Important questions then remain regarding what to assess, how to assess, and how not to overdo assessment, all while paying attention to correct technical skills. In this talk we will present our assessment plan and show how it evolved over time through repeated discussions to be more effective and efficient for both the students and the lecturers.

DIY Maths: Initial Findings on a Curated Learning Object

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This presentation examines the design, initial uses and development plans of a set of curated mathematics learning objects being developed at Unitec Institute of Technology, New Zealand's largest tertiary education polytechnic. Each learning object focuses on one small mathematics topic and contains a select assortment of freely-available mathematics resources, including instructional videos, self-grading quizzes, high-quality games and online tutors. The learning objects were intentionally designed to engage students and to allow for both social constructivist and cognitive constructivist modes of learning. The possible uses of this type of learning object are reflected upon along with initial student and lecturer feedback. The design framework of this object is potentially transferrable to many other learning contexts.

Beyond Statistical Methods: Teaching Critical Thinking to First Year University Students

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We discuss a major change in the way we teach our first year statistics course. We have redesigned the course with emphasis on teaching critical thinking. We recognized that most of the students take the course for general knowledge and support of other majors, and very few are planning to major in statistics. We identified the essential aspects of a first year statistics course, given this student mix, focusing on a simple question, "Given this is the last chance you have to teach statistics, what are the essential skills students need?" We have moved from thinking about statistics skills needed for a statistician to skills needed to participate in today's society. We have changed the way we deliver the course with less emphasis on lectures and more on computer based tutorials, Excel, and computer skills testing, and written assignments. Feedback from students shows that they are very receptive and enthusiastic.

Keywords: data cycle, self directed learning, formative feedback

Pre-Service Undergraduate Teachers' Pedagogical Content Knowledge

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Teaching the 'abstractness' of Mathematics to ensure students learn the concepts with conceptual and relational understanding is a great challenge to Mathematics teachers. Our pre-service undergraduate teachers' initial perception of teaching Mathematics is that with mastery of the Mathematics content, teaching Mathematics should be a breeze.

Pre-service undergraduate teachers' cognitive schemata and pedagogical content knowledge repertoire generally are less accessible, elaborate and interconnected than trained practitioners. In our curriculum studies modules, these pre-service undergraduate teachers are trained in the various pedagogies on our high school Mathematics content. As a proxy to actual classroom teaching, pedagogy content tests form part of the module assessment.

This paper is a preliminary study that reports on the pre-service undergraduate teachers' performance. They were assessed on their carefully thought-through plans in written communication. The emphasis is on the clarity and coherence in their written pedagogical presentations as well as their Mathematics proficiency.

The results are used to better equip these pre-service undergraduate teachers with the necessary repertoire of Mathematics pedagogical content knowledge and skills for more effective mathematics teaching and learning.

Key words: pedagogy, conceptual understanding, communication

Conceptions of Mathematics Among Engineering Students – A Puzzle in the Context of Identity

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Lecturers of first-year university mathematics often operate under the understanding that students enter mathematical studies with naïve conceptions of mathematics and that more mature conceptions need to be developed in the tertiary classroom. Students' conceptions of the nature and role of mathematics are pedagogically important as they impact on student learning and have the potential to influence what occurs in the classroom. As a subproject within a larger longitudinal study into the experiences at university of a cohort of students registered at the author's institution, engineering students' conceptions of mathematics were determined using a coding scheme developed elsewhere. The coding scheme suggested that the students choosing to study engineering exhibit an analytic and problem-solving conception of mathematics. A simultaneous analysis of student identity, as expressed primarily in interviews, suggests that these students do not embody designated identities specifically as engineers. Within the paradigm that development of engineering identity contributes to successful completion of an engineering degree, it seems that we, as engineering mathematics educators, have the opportunity to use the student's problem-solving conception of mathematics as a basis from which to develop other attributes of such an identity.

The One Variable Function Investigation as a Model of Exploratory Thinking

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Nowadays many technical skills needing earlier for engineering work are not essential more because advanced computer programs and so in teaching calculus in college of engineering we see as a main goal to promote deeper understanding of mathematical notions and creative exploratory thinking of the students. The investigation of the function is a complicated thought process, which includes recognizing main properties of functions, solving equations and inequalities, performing symbolic calculations, and using of analytical results in sketching of the function graph (M.Dagan, 2006). This process is very important for developing students' theoretical thinking.

Our experience of teaching calculus shows that many students see mathematics as an abstract subject only. They can do some formal symbolic operations successfully, but have difficulties in its interpretation. As a result, many students fail at the stage of syntheses of analytical results for construction of visual model (sketching of the graph). In order to reduce such difficulties and to give a real-life sense to the function investigation process, we use non-formal metaphoric schemes of function investigation, which assume answers on certain non-formal questions; and ask to give proper visual interpretations of each stage. The importance of metaphoric thoughts in teaching mathematics has been shown in many educational research studies (J.I. Acevedo, V. Font, J. Gumenez, 2003). We intend to present here our experience in this approach. Our practice reveals that the students in the experimental groups are doing the same task in a more self-supervisory way. They are more self-confident in their results and make fewer mistakes, not only in graphic representations, but also in formal operations, such as differentiation and calculations of limits. We observed the students of the experimental group correcting mistakes of such symbol operations, if they saw that these results were inconsistent with the graphical results of previous analysis. On the other hand, the students in the control group were trying to repair the sketch of the graph in order to make it more corresponding to mistaken analytical results. It should be noted that the students in the experimental group sketched the graphs of the functions more quickly than those in the control one.

Note, that we have considerable resistance from the part of educators and students as well. The educators' resistance was against doing something without simultaneous proof and the student's resistance followed from their belief that mathematics needs strongly determinate formal schemes of calculations only.

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Mayhem, Madness and Mathematics

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At the last Delta conference, I proposed an outline for a television game show that would highlight the creativity and fun of mathematics and statistics. The show is progressing, slowly but now with the help of the Delta conference delegates, we can fine tune and pit our wits to enjoy a fun-filled demonstration of Mayhem, Madness and Mathematics.

Be prepared for an abstract encounter of a real variable with irrational behaviour having no known limits that will integrate impropriety with exponential fun and primed frivolity.

Do we Trick or Train our Students in Maths?

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In 2010 a new extended degree programme was introduced at the University of Pretoria, the Engineering Augmented Degree programme (ENGAGE). Professional Orientation is a project based, skills-focused course comprising two semester modules in the ENGAGE programme. One of the projects, called the building project, comprises the application of three dimensional trigonometry to real life problems. In 2010, in order to prepare for the project, students had to complete an assignment on two and three dimensional trigonometry problems. In an effort to help students visualise a two dimensional diagram representing an object in three dimensions, students had to use an A4 sheet of paper, a pen and a ruler to fold a three dimensional model of a two dimensional diagram. The exercise proved to be far too difficult for students. Upon reflection, and to understand why students had such difficulty in performing the task, a follow-up assignment was designed where students worked in groups. The purpose of this assignment was to address misconceptions and guide students through the planning and problem solving stages. In 2011 the exercise was repeated, but this time it was carefully planned to include the planning and problem solving stages and was spread over two assignments. The first assignment dealt with solving two dimensional trigonometry problems and revisiting the definitions. The second assignment focused on solving three dimensional trigonometry problems, and then finally the folding of the paper model followed. This particular time the majority of the students performed well in the assignment.

Conceptually and Procedurally Oriented Mathematics Studies in Engineering Education: A Comparative Study

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This paper reports from a collaboration project between South Africa and Sweden with the aim to investigate whether the emphasis in undergraduate mathematics courses for engineering students would benefit from being more conceptually oriented than the traditional more procedurally oriented way of teaching. In the project we compare engineering students, lecturers and practising engineers in the two countries regarding their views on the role of mathematics in engineering education, with a focus on the distinction between conceptual and procedural knowledge. Differences in educational culture and organisation in the two countries may provide insights about each local situation, which would otherwise stay hidden, and similarities and differences may be found that can provide a theoretical basis as well as practical implications for how to organise mathematics education in engineering programs.

In this paper we report on results from third and fourth year engineering students from both countries and compare to earlier findings within the project from first/second year engineering students. An instrument, consisting of procedural and conceptual items as well as questions about students' views on the role of the different types of knowledge in their studies, was run with the target groups of engineering students at two universities, one in each country. We compare performance between these groups of students as well as their confidence in answering conceptual and procedural mathematics tasks. We also compare the students' conceptions on the role of conceptual and procedural mathematics problems within and outside their mathematics studies in their education, as well as about the relevance of these types of problems in their engineering studies.

In both countries and for both first/second year and third/fourth year students, the results pointed to a higher confidence in the students' performance on procedural tasks than on conceptual tasks, and the view that both categories of tasks are relevant for their engineering studies. However, the older students put more emphasis on the importance of conceptual knowledge than the younger students. These results point to a need to increase students' confidence in conceptually oriented tasks. Observed differences in outcomes were for example that procedural items are seen to be more common in their mathematics studies by the Swedish students while the South African students also find conceptual items common. Differences were also found concerning these types of mathematical tasks in their studies outside the mathematics courses. The mathematics courses as well as mathematics in other subjects within engineering education can thus be experienced differently by students from different institutions. These and other outcomes from the study have implications for how to reorganise the mathematics studies in engineering education.

Investigating Success Rates of First Level Statistics Students: A Case for Innovative Intervention

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The University of Pretoria is faced with the continuing problem of low success rates of its first year Statistics students. In 2005, while investigating the matric mathematics prerequisite/entrance criteria for Statistics, it was established that the 2004 pass rate was a disturbingly low 46.53%. This bleak picture has not improved dramatically despite continuing efforts on the part of the Department of Statistics to address the problem. Pass rates during the past five years have remained in the order of 60% and this continues to be a matter of great concern. Statistics (as opposed to Mathematical Statistics) is compulsory for all students who enrol for a BCom degree at the University, many of whom are not proficient in mathematics. This practice necessarily results in large student numbers, in the order of three thousand per annum. The number of students who qualifies to write the examination is obviously far lower than the initial enrolment in the course, for example last year (2010), 3 623 students enrolled while only 3 220 were admitted to the examination. This translates to an 11% drop in student numbers by the end of the semester, and should the pass rate be reported using the base, we are faced with a dismal 57%. A huge problem facing the lecturers is the large proportion of students, approximately 25% every year, who are repeating the course. The two specific Statistics modules for BCom students, comprising a first-level course, have been designated as so-called *high impact modules* and are thus targeted for additional tutoring support, serving as impetus for the current research. Major changes in the mathematics matriculation curriculum in 2008 have compounded the challenges faced by educators to improve students' performance and enrich the learning experience of students. Since 2006, various intervention strategies have been adopted by the Department of Statistics at the University of Pretoria to address the problem. This paper traces the effect of these mediations over the past couple of years, starting with a brief explanation of the set of problems unique to South Africa, followed by the impact of, firstly, compulsory homework assignments introduced in 2006. Using assignments as an aid to learning proved to be flawed and was hence replaced by a system of class tests in 2007. Lastly, an analysis is provided of students' performance before and after the introduction of the modified matric curriculum in 2008, to illustrate the impact of this change. A rationale is provided for a more interactive intervention where students will be required to complete assignments online - with immediate feedback - which will be implemented in 2012.

Mathematical Knowledge for Teaching: What do Secondary School Teachers Need?

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Internationally, there is a great deal of interest in describing and measuring the kinds of mathematical knowledge needed to teach mathematics effectively in primary and secondary schools. In Australia, this interest is paralleled by the development of national standards for professional knowledge that specify expectations across the years of teachers' careers. This presentation provides an opportunity for mathematicians and mathematics educators to consider the nature of mathematical knowledge needed by beginning teachers of secondary school mathematics. It is informed by a national project, funded by the Australian Learning and Teaching Council, which aims to improve the quality of university-based pre-service teacher education in mathematics.

The project involves seven Australian universities that offer pre-service teacher education programs. Online questionnaires were devised to measure two types of knowledge needed for effective teaching of mathematics: mathematical content knowledge and pedagogical content knowledge. The latter type of knowledge is important because it enables teachers of mathematics to recognise student ways of thinking, address student misconceptions about a mathematical concept, identify aspects of a mathematical task that affect its complexity, select appropriate representations of concepts, use a variety of examples and resources, and explain how mathematical topics fit into the curriculum.

The presentation will explain the process of creating multiple choice questionnaire items that aim to measure mathematical content knowledge and pedagogical content knowledge for teaching secondary school mathematics. It will invite discussion of alternative responses to selected items, share questionnaire data from pre-service secondary mathematics teachers, and invite participants to create new questionnaire items that probe mathematical knowledge for teaching in secondary schools.

3D Microworld Learning Environment for Discrete Mathematics

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We study to design Logo-based mathematical microworld environment equipped with 3D construction capability, 3D manipulation, and web-based communication for mathematics education. Extending the turtle metaphor of 2D Logo, we design simple and intuitive symbolic representation system that can create several turtle objects and operations. In our microworld environment, the symbolic representations constructing the turtle objects can be used for web-based collaborative learning, web 2.0 communication, and assessments.

Extending the paper “Representation Systems of Building Blocks in Logo-based Microworld” published in 2011 by Cho et al. at the Korean Society of Mathematical Education Journal *Research in Mathematical Education* **15-1**, 1-14, we have developed design principles of Logo-based 3D microworld in which the metaphor of ‘playing turtle’ is applied to construct 3D objects. We have used this microworld learning environment (<http://www.javamath.com/class>) for various mathematics classes including discrete mathematics class. We will figure out student’s ways to construct and represent combinatorial objects and operations in terms action symbols and 3D building blocks. We also present various mathematization activities applying the turtle objects and suggest the way to make good use of them in such mathematics education as discrete mathematics education, and will discuss the future application measures.

Using Maple TA™ in Undergraduate Mathematics

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Maple TA™ is an online teaching and assessment system, which uses the Maple™ mathematical computation software in grading student responses.

Several courses in the Mathematics Department of the University of Auckland use Maple TA™ to a greater or lesser extent. One course in particular has integrated it into its tutorial system. Results of student surveys from this course will be presented. In addition, our experience in implementing Maple TA™ and developing questions and assignments will be discussed.

As Maple TA™ is web-based, students who are enrolled in the courses which use it can access it either from the university computer laboratories or from elsewhere.

Mapping the Gap

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One of the aims of the Numeracy Centre at Macquarie University is to improve participation and completion rates in units with a mathematical/statistical component. The centre offers mathematics bridging programs at the University, a drop in service and additional classes for a number of first and second year mathematics and statistics units. We also run additional classes for the entry level chemistry unit and have run classes in the past for several units in the Faculty of Business and Economics.

In our contact with students we have noticed that there sometimes appears to be a disconnect between what students are expected to know and be able to do mathematically and the mathematical pre-requisites listed for their units. We were interested in whether this anecdotal evidence was reflecting reality and if so investigating possible solutions to any (mathematical) difficulties that students might be experiencing as they transitioned through their programs of study. In order to get a fuller picture we undertook a mapping exercise of the mathematical skills in units across the Science Faculty at Macquarie University. We were particularly interested in what academic staff expected their students to already know and the mathematical background of the students. We were also interested in finding out what mathematics students needed to learn or find out about in order to successfully complete their units and how that was being managed. This presentation discusses the results of this mapping exercise and possible solutions to the problems that were identified.

Enhancing Student Learning

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Conversations about the teaching of undergraduate mathematics are easily dominated by the recurrent themes of student preparedness, large student enrollments and pressure to produce satisfactory pass rates. A lot of undergraduate lecturers' time and energy is consumed in finding coping mechanisms as we deal with these issues.

In this talk I want to ask instead what we can do to enhance the learning experience, both for the student and the teacher. Framed by some of my own experiences, in particular in the use of technology and in exploiting the freedom of an academic career, we will discuss what "enhancing student learning" might entail and what benefit it can bring.

Effective teaching is centered on relationships. This begins with the relationship between teacher and student but extends to relationships with our subject, with technology and with peers and colleagues. Fostering these relationships is at the heart of an enhanced learning experience.

Delivering Mathematics Lectures With Tablet PCs: Lecturer and Student Reflections

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Tablet PCs are becoming established as powerful technological tools in the university classroom, particularly in the mathematical sciences. In this talk we reflect on student and lecturer experiences of the use of a tablet PC in undergraduate mathematics lectures at a South African university. Not only does the tablet PC transform the classroom, it can also be used for in-class communication and facilitates the electronic distribution of lecture notes and dynamic lecture screencasts, enriching the lecturer's and students' educational resources. Through student surveys we explore student perceptions of the technology, how they use the resources it helps produce and the lecturer's experience of the changed teaching environment.

A Model for Measuring the Quality of a Mathematics Question

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In this study, we develop a model, which we call the quality index (QI) model, for measuring how good a mathematics question is. Based on the literature on mathematics assessment, we develop a theoretical framework, using three measuring parameters: discrimination index, confidence index and expert expectation. The theoretical framework forms the foundation on which we form an opinion of the qualities and characteristics of a good mathematics question. The QI assigns a quantitative value to the quality of a question. We also give a visual representation of the nature of a question in terms of a radar chart using the mentioned criteria and illustrate the use of the QI model by applying the measure to question examples, given in each of two formats – provided response questions (PRQs) and constructed response questions (CRQs). A greater knowledge of the quality of mathematics questions can assist mathematics educators and assessors to improve their assessment programmes and enhance student learning in mathematics. A discussion on the application of this QI model to mathematics questions used in my secondary school teaching will be presented.

Having Fun With the First Year Engineers

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Transferable skills are usually known as an SEP (Somebody Else's Problem). At best we hope that our students pick them up by academic osmosis (i.e. sleep with the book under your pillow and in the morning it will have miraculously seeped in). Combine this with one of the university's largest courses which gets consistently terrible student ratings, has an even worse reputation amongst staff and a new lecturer that wants a research career and you have a recipe for certain disaster.

Experiences With Computer Aided Assessment in Large Mathematics Courses

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The School of Mathematical Sciences at The University of Adelaide has been using computer-aided assessment (via Maple T.A.) for both summative and formative assessment for the past six years. The significant advantages provided by computer-aided assessment (CAA), in terms of consistency, reusability and instantaneous feedback to students, make it a valuable addition to the teaching strategy for large mathematics courses. However it is not without its difficulties and limitations.

We will present a discussion of our experiences with a variety of uses of CAA in large first and second year mathematics courses, and how it ties in with more traditional methods of assessment, ranging from a replacement for human marking to a more comprehensive restructuring of ongoing assessment. A combination of student feedback, our own reflection and some relevant literature will be used to discuss the success of our implementations.

Population Regression Model in University Courses

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Statistics courses usually teach regression for samples rather than population and consider regression as fitting lines to data. In this talk we suggest that university courses in mathematical statistics include the population regression model at the beginning of the topic on regression. This approach has two benefits. The first is that it makes some of the ideas and proofs in regression clear because sample estimates are natural analogs of features in the population. The second is that this facilitates discussion of interesting applications such as the capital asset pricing model in financial mathematics.

We suggest using linear algebra to introduce the population regression model. Random variables with finite variances make a vector space, where a scalar product is defined by $\langle X, Y \rangle = E(X \cdot Y)$. In this space the distance between a random variable and its estimate equals a square root of the statistical mean-square error. So regression of a random variable Y to a random variable X becomes the function $f(X)$ closest to Y for all functions f from a certain class A . Therefore the regression of Y to X is just the orthogonal projection of Y onto the subspace $W = \{f(X): f \in A\}$. Different W 's produce different types of regression: general, linear, quadratic, polynomial, etc. The concepts of distance and projection involve students' geometrical intuition and help their conceptual understanding of regression.

Statistics textbooks use orthogonal projection to describe regression for samples, not population, and apply it to \mathbf{R}^n rather than the vector space of random variables. However, it is more logical to introduce the population regression model first and then derive the sample regression model following the common pattern in estimation theory when a population object is estimated from a sample.

In financial mathematics, portfolio analysis uses the population regression model to regress any portfolio x to the market portfolio m : $x = \alpha + \beta m + \varepsilon$. This linear regression model leads to the capital asset pricing model, which is one of the most important models in finance. The beta coefficients from the regression model are used for ordinal ranking of assets according to their systematic risk. This example can be used to demonstrate to students that the population regression model is a practical concept.

Mattetek, A Learning Platform for the Course Linear Algebra

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Mattetek is a project where we have developed a learning management system to support student learning in the course Linear Algebra. The course is an important base of an engineering education but also a course that many students perceive as difficult. We have developed an Internet-based course with a number of modules to support a reflective learning and support the students' understanding of relatively complex contexts. The course has the same content as the corresponding campus course and can be studied in parallel with, or separately.

The platform includes a number of modules. These modules are easy to change for use in other mathematical courses as well for other areas:

A **textbook**. Here is all the material in the form of the full theory with explanatory examples. For each chapter/sub-chapter, coupled appropriate exercises.

For each exercise task offers **tips and solutions**. These are organized so if the student needs help, he/she can use a menu to click through and get help on different levels. The tips are designed so that they can be used to understand the structures of the theory as an exercise task designed to illustrate. The predictions are intended to give the student a refined guidance through theory with such references to the textbook. In this subsection we make the first attempt to get students to stop and reflect on what they have done.

Following the recent exercises in each sub-chapter, the student is led to the so-called reflection exercises. These are the second attempt to get students to reflect and deepen their learning. Reflection exercises refer to capture the ideas contained within each sub-chapter, to using them to understand the larger contexts. These exercises are of the nature that students generally need to discuss these with other students.

Via two master theses in media technology (a master of engineering program), we have developed visualizations to illustrate key concepts such as projection, rotation, quadratic forms, etc.

To provide students with an enhanced feedback and motivation, the course has a number of self-correcting tests.

You will find the platform at

<http://wiki.math.se/wikis/samverkan/linalg-LIU/index.php/Huvudsida>

The language is Swedish and you need only standard programs to use it. At the talk there will be some demonstrations if Internet is working.

Using Video Clips to Support Student Learning

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The Numeracy Centre at Macquarie University publishes a series of books called the MUMS Modules. Originally designed to be able to be used for individual study in preparation for a range of mathematics and statistics courses at university level they are also used in our bridging courses. The books are available online as a set of PDF downloads. Taking advantage of technological advances over the last few years we have developed a series of short video clips that are linked to the examples in the online text. The clips have been created using a tablet PC and Camtasia. In this presentation we will describe and demonstrate our use of this technology and report on the feedback collected from students.

Academics' Perceptions of the Use and Relevance of Software in Mathematics, Statistics, Econometrics and Finance

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Software may be used in university teaching both to enhance student learning of discipline-content knowledge and skills, and to equip students with capabilities that will be useful in their future careers. Although research has indicated that software may be used as an effective way of engaging students and enhancing learning in certain scenarios, relatively little is known about academic practices with regard to the use of software more generally or about the extent to which this software is subsequently used by graduates in the workplace. This paper reports on the results of a survey of academics in quantitative and financial disciplines, which is part of a broader study also encompassing recent graduates and employers. Results indicate that a variety of software packages are in widespread use in university programs in quantitative and financial disciplines. Most surveyed academics believe that the use of software enhances learning and enables students to solve otherwise intractable problems. A majority also rate spreadsheet skills in particular as very important for the employability of graduates. A better understanding of the use of software in university teaching points the way to how curricula can be revised to enhance learning and prepare graduates for professional work.

The Changing Face of Mathematics Under the Melbourne Model

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In 2008, the University of Melbourne embarked on a new era in University Education in Australia, introducing the “Melbourne Model”. The Melbourne Model introduced six new three-year undergraduate degrees in Arts, Biomedicine, Commerce, Environments, Music and Science; which replaced all of the previous undergraduate degrees offered by the University. The new degrees provide both depth and breadth, as students are required to complete a major in one area of study, undertake subjects in three disciplines within their degree, and complete four to six breadth subjects outside of their degree. Professional qualifications are offered at a Master’s level where students graduating from any one of the six degrees, can choose from a variety of professional or specialist graduate programs.

Under the Melbourne Model, students wishing to major in mathematics and statistics enrol in a three-year Bachelor of Science degree. Students then have the option of a two-year Master of Science degree, which leads onto a three-year PhD degree.

The new degree structure required a complete overhaul of the undergraduate and postgraduate mathematics and statistics curriculum at the University as students are only required to take nine subjects to complete a major in mathematics and statistics. The breadth requirement has created many opportunities to teach a large number of students who historically have not taken mathematics at University. For the first time, we are teaching students from every one of the undergraduate degrees first, second and third year level subjects in mathematics and statistics.

In this talk we focus on the changing face of the undergraduate mathematics curriculum and teaching methods as a result of introducing the Melbourne Model. In particular, we discuss the changes required to the undergraduate mathematics and statistics major, the changing student cohorts and mathematics backgrounds of incoming students, changes in student enrolment patterns, service teaching opportunities for engineering, biomedicine and actuarial studies students, mathematics and statistics as breadth, as well as bridging subjects for students without Year 12 (final year) school mathematics.

Sharing my Experience in the Teaching and Learning of Mathematics and Statistics at Undergraduate Level: Challenges and Mitigation Strategies

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The teaching of mathematics and statistics at undergraduate level forms the cornerstone of many disciplines such as engineering, the built environment, computing sciences, and many other natural, behavioural and social sciences. Both mathematics and statistics are grouped into two main categories namely theory (or pure) and applied. The majority of undergraduate students have a perception that theory which is also referred to as abstract mathematics/statistics is ‘difficult or hard’. Experience through years of observation, summative assessments and students’ feedback has indicated that students who have been exposed to applied mathematics/statistics often find it hard to appreciate abstract mathematics/statistics in their penultimate year. Students also appreciate a mathematics/statistics concept better when examples given are related to their field of specialisation. It has been found effective that when a facilitator/lecturer is introducing a ‘hard’ concept he/she must start by giving the simplest example or real world application of the concept. In this study particular attention is paid to teaching and learning approaches that are aimed at maximising student participation, creating interest and motivating them to believe in their own potential. An outline of the common students’ challenges as well as mitigation strategies in the teaching and learning of mathematics and statistics at undergraduate level is presented.

The Illustrative Mathematics Project

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The recently developed Common Core State Standards for Mathematics in the U.S.A. define the skills, understandings and practices intended for children to learn as they progress through their schooling. As states develop assessments, curricula, and professional development based on the standards, and as teachers use them in the classroom, they will benefit from examples of student tasks that develop and assess those skills, understandings and practices. The ultimate goal of the Illustrative Mathematics Project is to develop a complete set of such tasks for each standard, ranging from easy to difficult, and from simple illustrations of single standards to complex tasks spanning many standards.

The project will guide the work of U.S. states, assessment consortia, and testing companies by illustrating the range and types of mathematical work that students will experience in a faithful implementation of the standards. The process of selecting and reviewing tasks will pay attention both to what is known about how students think and to mathematical structure. A complete set of tasks will provide a reference map for assessment, curriculum design, and professional development.

For the sample tasks to achieve this goal they must have the respect and attention of a diverse group of stakeholders; state and district supervisors and assessment experts, assessment consortia, teachers, mathematics education researchers, mathematicians, textbook and testing companies. The project will build that respect and attention by

- developing a process for error-checking and vetting tasks to ensure that they are mathematically correct, pedagogically appropriate, and of the highest possible quality
- building a social network through the project website consisting of teachers, mathematicians, and educators who have the opportunity to discuss tasks, build their own “playlists” of tasks, and submit tasks to the vetting process.

Thus the vetting process will serve as a bootstrapping process for building a sense of professionalism around task design.

A prototype of the web tool for viewing standards and tasks is at <http://illustrativemathematics.org>. The tool will allow users to arrange standards in various ways and view tasks associated not only with particular standards, but with streams of standards and other higher order structures.

What do we Want Students to Learn About Mathematics in Preparation for University - And how can we Help Them Learn it?

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At the level of national policies, much of the current discussion on mathematics education centres on the mathematics that students *need* to know (for college or career readiness, for example). It will be argued in this talk that an overly strong emphasis on need is both limiting and dangerous, and that we should be asking instead what mathematics we *want* students to learn. Examples of the latter will be given, and the distinction between the two approaches will be illustrated with references to national standards documents, university course syllabi, and surveys of university faculty.

Learning Difficulties With Solids of Revolution

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This study reveals some of the challenges that students at one FET College in South Africa experienced when they are faced with applications of the definite integral, such as solids of revolution. Many students are not fully competent in drawing graphs and in interpreting the region bounded by the given graphs. Even if the graphs are drawn, an obstacle is that students do not know how to select the strip that is used to approximate the bounded region, stemming from the Riemann sum. Although many students are able to deliver the correct formula to calculate the volume, be it disc, washer or shell, as was emphasised in the classroom, they struggle when drawing the 3D representation of the rotated strip and the solid of revolution generated. Students seem to succeed with tasks requiring simple manipulation skills and do not succeed if the tasks require a higher level of cognitive development, including selecting the strip and rotating it. What is seen to be the major problem is that the students prefer to do manipulations and avoid using visual skills in interpreting the drawn graphs and the strip that is being rotated.

Essential Characteristics for Engineering Mathematics Learners

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The aim of this presentation is to discuss some positive and negative attributes of mathematics adult learners in a research project. In recent years, tertiary institutions have faced increasing pressure to raise retention and success rates of certain programmes, in order to meet the course completion and student retention policies, which are implemented by the New Zealand Tertiary Education Commission (2010). In these institutions, mathematics has been perceived as a ‘gatekeeper’ in the engineering departments because it is a compulsory first-year subject. Hence, students are required to pass mathematics before enrolling in other engineering courses. Thus, over the years, the departments attributed the problem of low retention of engineering mathematics students to the students, who were inadequately prepared for tertiary mathematics. It is in this context that the project was carried out. The research was conducted with a cohort of engineering students in a local New Zealand polytechnic from February to June in 2010. Students ($n=22$) were given an algebra test, which assessed their algebraic knowledge and skills at the start of the course in February. In March, the sample group completed a closed-form questionnaire that investigated perceived levels of self-concept, interest, enjoyment and mathematical usefulness. Subsequently, correlations between student cognitive abilities and their motivation (interest, mathematical usefulness, enjoyment and self-efficacy) in learning mathematics were analysed. In addition to the quantitative analyses, qualitative interview protocols (Usher, 2009) based on Bandura’s (1986; 1995) theory of self-efficacy were employed. In April, the student interviews resulted in *six* case studies of mixed-ability (low, average and high) learners, chosen from the initial test sample, in order to capture their personal, social, temporal and situational conditions under which they appraised their beliefs and experiences. The primary findings were that even though some adult learners, who were predominantly mature students (older than 20 years old), may possess poor mathematical knowledge, it was important that they displayed strong self-concept, interest and enjoyment in learning mathematics, and perceived mathematics to be a useful subject in order to succeed in mathematics. Furthermore, those who succeeded in the course coped well under pressure and received some form of academic support. In the context of adult learning, we will discuss the attributes of successful students, the role of the first algebra test as a possible predictor of the final examination results, and the sources of self-efficacy and other motivational factors, such as greater interest and meaningfulness in learning mathematics and the importance of fostering effective study skills. Lastly, some limitations of the project will be explored for further research.

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Jumping Over a Gap as the Base of Learning

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Many people point out the huge gap between school math and university math and there are many who have endeavoured to help their students to pass over the gap by giving their lectures new styles more attractive to students or more supportive for slow learners of math, but there are not many who point out the fact that there are gaps also between elementary level math and secondary level math. There is a gap even between junior high school math (K7-9) and senior high school math (K10-12). Despite the gaps many students appear to have been able to jump over them with less difficulty before their tertiary stage.

Almost all the gaps correspond to the revolutionary changes in the history of mathematics. For example the concept of integers including zero and negative numbers is the first challenge for students in their first grade of junior high schools (K7). History of mathematics tells us that there were famous scholars even in 18th century who argued against the concept of negative numbers. In fact, if we base ‘purely’ upon the concept of cardinal numbers as is most natural in the first primitive approach by humankind toward numbers, it is extremely difficult to conceive negatives. The embarrassment which most students encounter at university calculus corresponds to the revolutionary change of mathematical paradigm which took place in 19th century, later named as the ‘arithmetization of analysis’. Many professors complain about the lack of understanding even of the most fundamental concepts like differentiation, but history of mathematics tells that the great mathematicians discovered the most profound truths of calculus without the very definition of differentiation or of integration. In this sense it is quite natural that the development in individual understanding is well compared to the historical development of human knowledge, which is often quoted based upon the insight from genetical sciences.

Most university students should be regarded as ‘conquerors’ of huge gaps they have met before university days. This means that learning or acquiring innovative knowledge is nothing but jumping a leap, bigger or smaller. Therefore we should search a real reason why the ‘final’ gap appears to be too greater for ordinary students to jump over. I will suggest some points which I think are the most essential to ‘bridge the gap’:

- 1) The big influence of school education upon the way of understanding mathematics: too much emphasis put upon problem solving in Japan.
- 2) The lack of liaison between schools and universities: even if understanding of tertiary level mathematics by school teachers might not be indispensable, school teachers can teach their students about theoretical shortage of school mathematics and encourage them to study further in universities. University teachers should be more tolerate for students’ difficulty to overcome their ‘brain washed’ way of understanding mathematics.
- 3) Various contributions are of vital importance to induce students to be self-conscious of the leap; e.g. presentation of good (counter) examples that cannot be understood in the standard framework of high school math.

Using History to Deepen Teachers' Understanding of a General Positional Notation

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In this presentation I will describe and discuss a teaching sequence that I developed as part of a research study for a deep understanding of the structure of the place value numeration system. This framework was developed out of ideas extracted from an analysis of the historical development of the current Hindu-Arabic place value system (Datta & Singh, 2001), and viewed in light of educational research.

Although the decimal place value system is vital to progress in mathematics, particularly in arithmetic and algebra, teachers and researchers (e.g. Fuson, 1990) acknowledge that despite instructional efforts, many students have difficulty in understanding its multiplicative structure. Research suggests that teachers' (inadequate) conceptual and content knowledge (e.g. Zazkis & Khoury, 1993) may be a contributing factor. Teachers' familiarity with the base-ten number system, while helping them to *use* it, however, can prevent them from fully comprehending the difficulty students have in trying to understand this abstract concept. In addition, teachers may be unaware of some of the components and subtleties that comprise it and this impacts their practice; this insufficient (or lack of) awareness is passed on from one generation to another. What is required is a depth of understanding for *teaching* the system and teachers' own understanding is crucial to help all students understand the place value structure.

In view of the above, I propose to present evidence for the need for professional development of teachers with respect to place value numeration and will also present a framework for the teaching of general positional notation which was tested with secondary students (and yielding positive results). This teaching framework was developed from an analysis and adaptation of the broad historical stages (such as naming and working with large numbers) that the system went through in its evolution, as well as the phases (such as the additive and then multiplicative) within the written numeration system, which may aid a deep understanding of place value notation. My presentation will also include a module of work linking multiple representations, powers as repeated multiplication using concrete materials, and extending this to multiple bases for generalising the idea of the place value system. Thus, generalisation of place value notation could provide a pathway for beginning algebra and hence this presentation may be useful for all teachers, including secondary teachers.

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Building Leadership Capacity in the Development and Sharing of Mathematics Learning Resources Across Disciplines and Universities

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This presentation will provide an overview of a project funded by the Australian Learning Council.

This project involved developing human resources, aligning objectives with the needs of an institution and addressing the legal and technical issues to allow Australian academics to share teaching and learning video resources. Through symposia, workshops, meetings and engaging with the wider academic community, participants have been able to develop their skills in tablet technology, create resources and deploy them in new learning designs. These resources and learning designs have been associated with improved learning outcomes for students. The presentation of issues related to resource creation, video genres, evaluation and learning designs in many venues has inspired others to become involved.

The sustained hosting of a collection of resources has been possible through the alignment with core university infrastructure at the host institution, the University of Wollongong (UoW). The resources are available through Content Without Borders <<http://oer.equella.com/access/home.do>>. To minimise legal risk associated with hosting resources from other institutions with diverse intellectual property rights a Memorandum of Understanding between the host institution and contributing institutions was developed. Several Australian universities are now engaged in the process of completing and at times negotiating the refinement the Memorandum of Understanding in order that their staff may contribute to the Share World collection.

Keywords: open access; student learning; table technology; video resources

Algebras, Lattices, Varieties

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For making human ideas explicit in mathematical models, we suggest structures from Contextual Logic, a mathematical theory of human thought. Already Kant emphasized that the main functions of human thinking are concepts, judgments, and conclusions. Humans grasp realities first by concepts, then combine concepts to judgments and conclude judgments from other judgments. Therefore, traditional philosophical logic was based on doctrines of concepts, judgments, and conclusions. Contextual Logic mathematizes this understanding and, in doing this, elaborates tight connections between language, logic, and mathematics. Philosophically, our approach is based on Peirce's pragmatism. Examples shall demonstrate how modellings can be performed by concept lattices and concept graphs, the basic structures of Contextual Logic.

Key Words: modelling with concepts, lattices and concept graphs

Exploring a DVD Driven Approach for Teaching and Learning Mathematics, at Secondary School Level, With a Framework of Blended Learning

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The demise of apartheid brought with it great expectations of a transformed education system that would deliver a quality education for all South African citizens. However post-apartheid South Africa is still experiencing many challenges at secondary school level especially with regard to mathematics teaching and learning. The education system has failed to deliver and universities are concerned with the under-preparation of learners for tertiary studies especially in mathematics. There is a shortage of suitably qualified mathematics teachers and it has become imperative for universities to address the shortcomings of such an impasse. It has become common practice over the last few years for universities to initiate augmented programmes in mathematics for secondary school learners in surrounding areas. Through this practice, it is hoped that, to some degree the question of learners' under preparedness with regard to mathematics could be successfully addressed. This study describes one such augmented program, a particular approach of blended learning, devised for the Incubator School Project (ISP), an initiative of the Nelson Mandela Metropolitan University (NMMU) in the Eastern Cape of South Africa. The defining feature of this blended environment is that it incorporates DVD technology, which offers an affordable and accessible option for the particular group of learners and the schools they attend.

The participants comprised 147 learners and facilitators of the 2009 ISP. This case study explores the particular blended approach and reports six fold on the approach – qualitatively based firstly on a questionnaire completed by learners and secondly on interviews of learners, thirdly on the facilitators reports, fourthly quantitatively on learner performance before and after the intervention. Fifthly six schools are used as a case study where the mathematics performance of the learners who participated in the ISP is compared to those who did not participate in the ISP. Finally the scope of blending of this model is evaluated by means of a radar chart, adapted from an existing radar measure.

This research revealed that using the DVD approach within a blended learning environment did lead to an improvement in learners perceptions about mathematics, an improvement in the manner in which they learned mathematics, an extension in their mathematics knowledge and provided learners with a supportive environment in which to learn mathematics. The findings suggest that this approach impacted favourably on the mathematics learning and performance of these learners. It is hoped that knowledge gained from this study may contribute to alleviating the mathematics crisis experienced in this country by describing an implemented blended learning intervention and presenting its pitfalls and successes.

Sampling Theory, Survey Methodology or Survey Theory?

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Teaching students about how to design, run and analyse surveys can be seen as important given the number of surveys being run and reported on, but also it has the advantage that the topic can be readily seen by students as useful practical knowledge. However there is the issue of what should and can be taught. Sampling theory is a major part of what is taught at university, but the practice of surveying requires knowledge of survey methodology which is often seen as more a bundle of techniques with some theory here and there. However many of the methods require good knowledge of statistical theory and should be taught along with sampling theory as a coherent and statistically based theory of surveying.

How Individual Personalities Affect Achievement and Behaviour

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Session will provide insight into how individual teacher's and student's personalities affect 1) teacher behaviour, 2) student behaviour, 3) student achievement, and 4) teacher/student interaction and relationship. Presenter has conducted professional development workshops with STEM teachers since 2006, which has resulted in a significant improvement in teacher satisfaction, student self-efficacy, student achievement, and student behaviour. Teacher retention and student achievement can be positively affected by an understanding of how individual personalities think, feel, learn, and behave. Information presented will provide foundational knowledge that is intended to lessen mathematics anxiety by creating an environment in which teacher/student understanding is deepened and enhanced. Participants will discover their own personality type during the workshop and learn how their personality determines how they perceive and react to external and internal stimuli. An overview of the four major personality types will be presented including strategies for the classroom along with qualitative and quantitative data from previous STEM personality workshop evaluations.

Keywords: STEM Professional Development, Personality Styles, Learning Styles

Ironies in Mathematics Professional Development

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Which teachers pick up good ideas and run with them? Sometimes it is not those to whom the professional development is targeted. In the oral presentation we will recount our experiences over a seven-year project with a high school mathematics and science program. The initial focus of the technology-based interventions was the mathematics department. The University faculty and staff worked intensely in genuine partnership with the department to develop a new program that included graphing calculators for all students, a reform-based, activity-oriented mathematics curriculum, and a program of lesson study to build professional learning communities in mathematics. After seven years, the effects of ubiquitous technology are clear; students and teachers expect to have electronic tools available at all times. Yet, mathematics instruction, as documented by classroom observation, remains teacher-centered. We will discuss how and why this happened and note a powerful event in *legitimate peripheral participation* (Lave & Wenger, 1991); the high school science department, largely independent of University targeting, but taking full advantage of University support, developed their own technology-rich, student-centered curriculum changes that continue to evolve. These include case study approaches in physics, the adoption of an NSF-developed reform curriculum and the “flipped classroom” in which lectures are online and classroom time is devoted to problem-based interactions. Our presentation will note the specific events of this transformation in light of the larger trends in American education and highlight examples from the research literature on professional development. We will also note the changing role of the University collaborators from providers of research expertise to support personnel for professional development. This surprising change is a role reversal between the professional developers and the teachers who are receiving the professional development wherein the teachers are now taking the lead. At the same time, those initially targeted for change, have found the *dynamic conservative* (Schön, 1971) mechanisms maintaining the status quo in mathematics instruction.

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The Mathematics of Assessment

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For many years online assessment of mathematics has developed in scope, sophistication and extent of use. In the last ten years computer algebra systems have increasingly been used to support online assessment. Typically a student will be asked to answer a mathematical question through a web browser. Their answer takes the form of a mathematical expression, which includes equations, sets, lists or matrices. The system then assesses their answer and provides feedback in the form of a numerical mark or text based formative feedback. Other outcomes include notes for statistical analysis, or to build a profile from which adaptive testing algorithms may choose subsequent questions. There are many examples of CAA systems able to do this, including the STACK system designed by the author. Text-book publishers are increasingly providing online exercises to accompany traditional books.

This talk will provide a survey of the interesting mathematics associated with online computer aided assessment (CAA) of mathematics. The mathematical issues we address include randomly generating feasible problems, and establishing the properties of students' answers to provide feedback. The desire to automate a process throws into sharp detail the ambiguities and inconsistencies which traditional assessment might tolerate. For example, what does the word “simplify” mean, or what does it mean for two mathematical expressions to be “the same”? Put another way, in what variety of senses can two expressions be considered the same or different? We will also examine the theoretical and practical limitations in establishing this automatically with computer algebra.

As a specific example we will examine the mathematics of systems of polynomial equations, with a particular focus on establishing when two sets of polynomial equations represent the “same” situation. Such systems of equations arise naturally when asking students to model situations and when assessing the answers to algebra “story problems”. By calculating the polynomial Gröbner basis we will be able to establish whether the student's system is correct. We may also decide whether their systems of equations is inconsistent, underdetermined or overdetermined. In each case we can isolate the equation/equations responsible and provide feedback. Examples of specific problems, and students' reactions, will be presented.

These issues clearly concern designers and teachers using CAA. We argue they should also impinge on teaching more generally, since they strike at the heart of meaning in mathematics. Ultimately, we argue, students would gain valuable insights into our subject through an appreciation of these issues. This talk aims to raise awareness of, and appreciation for, the subtle issues associated with the mathematics of assessment.

Tangible Models in Teaching of Calculus

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There is a growing tendency of the last years to use more and more computer illustration of mathematical concepts and facts in teaching mathematics. Computers certainly have great options in this matter. But we think that computer models should not replace completely the real tangible models in mathematics teaching. We want to emphasize that tangible - it's not just visual, visual - it's not just what student can see on the picture or on the computer screen, but what he can see and touch physically and hold in the hands. It is also important to encourage students to make an appropriate models by own hands by using of a bound materials. To the famous aphorism "it is better once to see than seven times to hear" we want to add "it is better once to make itself than seven times to see".

Here we intend to show our long-term experience of using of real models in teaching of Calculus of function of several variables for engineering students. For example, the simple real model for demonstration of distance between two skew lines and using of it for finding idea to analytical solution of the problem, models for the better understanding of gradient and directional derivative, tangent plane (M. Dagan, P. Satianov, 2002) and so on.

We note that our approach based on Galperin's theory about the phased formation of mental activity and Shepard's attitude toward psychological aspects of the studies of three-dimensional objects and the mental actions connected with this process.

Our experience shows that such approach improves the understanding of an abstract notions and facts of Calculus. It also promotes creative engineering thinking of the students and stimulates their interest in learning of Calculus and its application.

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Teaching Mathematics in the PC Lab – The Students' Viewpoints

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The compulsory course in matrix algebra at the Faculty of Business and Economics at Schmalkalden University is no longer given in a traditional classroom setting (using blackboard, overhead projector, and pocket calculators) but in the PC lab (one or two students in front of a PC, instructor's PC connected to a projector), where the Computer Algebra System DERIVE is used throughout the semester. The faculty's DERIVE license also covers the private PCs of the students such that they can use the software at no charge at home as long as they are enrolled at the faculty. Students also have access to DERIVE during the final exam in the PC lab (here, naturally, only one student per PC). While the course and the examination have taken place in the PC lab for a number of years, for the first time a survey was carried out in Oct 2010 to find out if the students prefer a traditional or a technology-based course, and how well they cope with the technology. Results from this survey will be presented.

Mathematics in Gilbert and Sullivan

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This workshop/presentation will describe the mathematical references in seven of the 14 comic operas created by the Victorian team of Sir William Schwenk Gilbert and Sir Arthur S. Sullivan between 1871 and 1896. Gilbert had many talents – lawyer, poet, lyricist, artist – but he also had engineering training in preparation for military service, and thus was familiar with the academic mathematics of his day. The presentation will borrow from “Cheerful Facts about Matters Mathematical,” a recent scholarly paper by mathematician (and fellow G&S enthusiast) Tom Drucker, presented at the International G&S Festival in Gettysburg PA. But it will not emphasize the scholarly aspects. Some of the songs will be performed live, and the audience will be encouraged to join in the choruses.

Model Eliciting Activities – A Teachers’ Perspective

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In many mathematics classrooms students are taught mathematics skills first, then are asked to apply their knowledge in application problems. Our frustration with the efficacy of this pedagogy along with a desire to engage students in a deeper level of mathematical thinking has led us to explore Model Eliciting Activities (MEAs) [1]. MEAs are context-rich mathematical modelling activities that encourage students to explore and create their own mathematical models while solving open ended tasks. Over the past 18 months, we have been involved in creating and implementing MEAs for a Foundation Studies course in a tertiary environment. In this short presentation we will share our experiences, the possible benefits for learning, and difficulties with implementation MEAs.

There are six principles which govern the design of MEAs: model construction, relevance, generalisability, self-assessment, documentation and elegance. Students are asked to create a mathematical model to solve a problem set in a context selected to be relevant to them. There are no restrictions on the method or materials students can choose to explore the problem, and they draw on their own knowledge and problem solving skills. As a result, for students to engage in mathematical thinking, the MEA task must intrinsically require a mathematical model to be solved. The model is then generalised so that it can be used to solve other similar problems. Students check that their solution works through self-assessment, and produce a document explaining their solution.

During implementation the teacher’s role becomes that of a facilitator, allowing students to work in groups to explore and mathematise the problem. MEAs can be a daunting prospect for both students and teachers alike and have been nick-named “scarily open problems”. Challenges for teachers include the change in the teacher’s role in the class, and the need to follow a wide variety of students’ thinking. Students will be challenged by the task itself, but may also be unfamiliar with group work and inexperienced with open tasks.

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A Content-Focused Professional Development Project for Teachers of Mathematics From Design and Implementation Through Evaluation: A Project Director's Perspective

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The West Texas Middle School Mathematics Partnership (WTMSMP) is a partnership consisting of four institutions of higher education (IHEs), an independent school district and three Texas Educational Service Centre Regions (ESCs). The partners serve 15,168 teachers with 199,584 students spread over 84,000 square miles of West Texas. The ultimate goal of WTMSMP is for the student population of this region, comprised of a large percentage of disadvantaged and Hispanic youth, to reach higher levels of mathematical achievement. This will be achieved by increased interactions with math teachers who possess a deep conceptual understanding of elementary mathematics, have a strong belief in their ability to teach mathematics to diverse student populations, and have the self-determination to influence colleagues and administrators.

This presentation will provide information concerning the design and structure of the project, including its staff and research team, the teacher participants and their school districts, and the students it hopes to impact. The project evaluation design as well as related research and preliminary results will be discussed. Further formative information will be provided through anecdotes from participating teachers who have benefited from this program, and the challenges faced including measuring student achievement and participant motivation.

How do Undergraduate Mathematics Students Want to Learn?

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It seems natural that students should have the most buy-in for courses that reflect their desires, incorporating things they want to do in the ways they want to do them. It also seems natural that students should be most successful in their studies when they are engaged in them; that is, when they have the most buy-in. Therefore, if it can be determined what students perceive as an ideal learning experience in an undergraduate mathematics degree, then this can be used to shape undergraduate mathematics degrees accordingly, leading to more enthused, successful students.

This oral presentation addresses the question of what students perceive as an ideal learning experience in an undergraduate mathematics degree. To do this, opinions will be presented which have been obtained through open discussion with third-year mathematics students at the University of Canterbury, as well as reflections on my own three years of undergraduate study to date. Due to the large restructuring of course delivery in Semester 1 of 2011 caused by the earthquakes in Canterbury, I and my classmates have been exposed to more styles and mechanisms of teaching and learning than may usually be the case, particularly with regard to online instruction. As a result, we are well-positioned to make comments which are valuable for the future direction of undergraduate educational experiences in mathematics.

The emphasis of the presentation will be on styles of teaching and learning, delivery mechanisms and related concepts, rather than specific curriculum content. However, reference will be made to areas of study which students particularly enjoy and the reasons for this. The presentation will also give some consideration to the reasons for which students choose to study mathematics and the things which motivate them in these studies. The ultimate goal is to present an image of the student's ideal undergraduate mathematics degree, one which aims to produce larger amounts of buy-in, and correspondingly higher degrees of success, from students.

Thoughts Expressed by Teachers and Learners on Factors that Facilitate Learners' Performance in Mathematics in South Africa

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The purpose of this study is to advance the understanding of why some mathematics classrooms in disadvantaged communities in South Africa are successful and others are not. The participants in this research were educators and learners from historically disadvantaged schools from similar socio-economic backgrounds. Ten rural schools participated in the study, all government schools, with schools selected on the basis of their accessibility and performance. They represent learners from both high performing (HPS) and low performing schools (LPS) in mathematics. Data was collected from learner focus group interviews and individual teacher interviews. Thoughts expressed by educators and learners from high achieving schools and from low achieving schools are juxtaposed and point to factors such as learners' and teachers' commitment and motivation; attitudes and self-concept; learners' career prospects; learners' perceptions of peers and teachers; and teachers' perceptions of learners. These factors appear to influence disadvantaged learners' decisions to persist and achieve in mathematics in spite of their difficult circumstances.

Problems Encountered in Teaching Mathematics to Undergraduates in Kenya

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Many University Lecturers of Mathematics look back to the period before 1990, as a golden era of University teaching, when the students were well prepared for a rigorous Degree Programme, and would work hard, and enjoy their Programme. These students had passed the Kenya Advanced Certificate of Education (KACE). However the admission of students holding the Kenya Certificate of Secondary Education (KCSE) from 1990 has presented problems for University Lecturers, especially in Mathematics, where entrants are younger, less mature, and not so well prepared for their Degree Programme. The paper seeks to identify the major problems, mentions steps taken to solve these problems, and suggests further steps that should be taken to reduce these problems.

Numerical Investigation Into the Existence of Limit Cycles in Two Dimensional Predator-Prey Systems

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The original Lotka-Volterra model does not possess any limit cycles. In recent years this model has been modified to take disturbances into consideration and allow populations to return to their original numbers. By introducing logistic growth and a Holling-Type II functional response to the traditional Lotka-Volterra-type models, it has been proven that for such systems a unique, stable limit cycle exists. The proofs make use of Dulac functions, Liénard equations and invariant regions, relying on work done by Poincaré, Poincaré-Bendixson, Dulac, Liénard, Cheng, Zhang, Cherkas and Zhilevich, Waltman, Rosenzweig and MacArthur and others and are generally perceived as difficult.

In this paper the population model

$$\begin{aligned}\dot{x} &= rx\left(1 - \frac{x}{K}\right) - y\left(\frac{ax}{b+x}\right) \\ \dot{y} &= -cy + dy\left(\frac{ax}{b+x}\right)\end{aligned}$$

is used to review some of the classical analytical methods applied when proving the existence of an unique, stable limit cycle.

Computer Algebra Systems (CAS) are ideally suited to illustrate the existence of limit cycles in nonlinear systems. The model in question is used as a vehicle to introduce a simple numerical algorithm to confirm that a single stable limit cycle exists.

So, Just What Should a Mathematics Degree Consist Of?

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In the UK, the proportion of school leavers going to university has now reached 50%, and the number of institutions awarding degrees has risen to over 300. The graduate job market is tough, especially during a recession, even for those with a degree in mathematics, and hence graduate ‘quality’ is more important than ever.

One result of this is the increasing attention given to the new measures by which the performance of Universities and of individual programmes can be judged, such as the National Student Survey, the DLHE survey of graduate employment, league tables of performance and, from next year, Key Information Sets.

A further result is the increasing importance attached to the additional skills students should be gaining at University, over and above their subject-specific skills. With large rises in tuition fees being phased in from 2012 in England, many applicants will be concerned - amongst other things - that courses provide them with the full range of skills necessary to successfully gain graduate level employment. This is especially true at a time of high levels of graduate unemployment. The Guardian, for example, recently reported that “20% of recent graduates are unemployed – the highest proportion for a decade” and “Almost half of all recent graduates believe their university education did not adequately equip them for the world of work”

All of this raises a fundamental question about what Mathematics degrees should contain, in particular, should they include ‘employability’ skills. If so, to what extent does the current curriculum (and the learning, teaching and assessment strategies that deliver, support and assess it) incorporate these principles - and where it does, how successful is it?

Commissioned by the Mathematical Sciences Curriculum Innovation Project as part of the National HE STEM Programme, I have coordinated a project gathering a series of short case studies, written by staff in Mathematics departments in a variety of HEIs, each focussed on specific graduate skills and providing examples of ways in which these have been successfully developed. The project also aimed to evaluate what techniques have been successful and why, and to make some suggestions for how they may be used elsewhere. The final booklet was published in April 2011, and workshops were held in five of the HE STEM regions during April and May. It is available electronically from <http://maths.shu.ac.uk/MSOR/GraduateSkills/>.

In this presentation I will describe some of the results of the project, hopefully provoking a discussion on the fundamental question raised in the title.

MapleTA Assessment: Earthquake Accelerations

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The University of Canterbury began using MapleTA as an assessment tool in our first year undergraduate mathematics courses during the summer semester of 2009-2010. Initially we used this software for summative assessment only, but we began to discuss making greater use of the software by developing learning modules to encourage student engagement and improve learning outcomes. This move was accelerated by the February 2011 earthquake in Christchurch. With very few computer labs available, supervised skills tests were dropped in favour of weekly learning modules to support student learning in the post-quake environment. Student responses were gathered in a survey taken at the end of the semester, and together with MapleTA and course results, we have a large amount of data to analyse. This talk will summarise our experiences with MapleTA so far and present initial findings from this data.

Identifying Major Concepts in Two Second Level Undergraduate Mathematics Courses

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What do we consider to be the major mathematical concepts, ideas or skills that we want to teach students in our undergraduate mathematics courses? What are the factors that influence the choice of these concepts, ideas or skills?

Twenty one lecturers were interviewed about twenty three courses from four levels of undergraduate mathematics and statistics. In each interview the lecturers were asked to identify the major concepts that they wanted their students to gain by the end of the course and their reasons for choosing those concepts. In this talk I follow on from a paper entitled “The big ideas in two large first level courses of undergraduate mathematics” which I presented at the AAMT-MERGA conference 2011. From my interviews with lecturers of first level courses, it was clear that teaching mathematics goes beyond teaching mathematical concepts alone. Some lecturers identified ideas such as critical thinking and mathematical confidence as major ideas that they wanted the students to gain. The interviews I will be discussing in this talk were conducted with two lecturers from different second level mathematics courses. Responses from these lecturers demonstrate how in second level courses the emphasis for teaching moves more towards the mathematical concepts themselves, as opposed to general skills. These two courses were chosen for discussion as they represent different influences on the choice of major mathematical concepts by the lecturer. In one course, the lecturer chose the important mathematical concepts based on the course content. In the other course, the lecturer chose the important mathematical concepts based on both the previous knowledge of the current students as well as the course content.

The concepts will be discussed in terms of their *concept definitions* as they are formally described by the discipline and their *concept usage* if they are to be applied. Those concepts with the additional feature of transforming a student’s thinking will be discussed in terms of the theory of threshold concepts.

The Negative Spiral of Mathematical Education in Japan

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In the observation I have done in Japan as a mathematical student, the reason why most students encounter difficulty to understand university mathematics lies not in the differences of the level of rigour or abstractness, but in the “cultural difference” around the art of understanding mathematics. In short, solid understanding of foundational concepts is the key of university mathematics, while technical knowledge about problem solving is regarded as most essential in high school mathematics.

A gap between high school math and university math can be easily pointed out in Japan as well as in other countries but I want to discuss a peculiar character of Japanese gap in several aspects. Generally speaking math teachers do not love mathematics, because they experience serious difficulties in understanding tertiary level mathematics, which appear to them completely different from the high school mathematics they were fond of.

In the severely competitive environment of young people for better schools, which starts even in kindergarten age, people are generally obliged to consider mathematics as the most important subject and most young people come to desire to be better in mathematics to be winners in the race. Parents hope school teachers to instruct their children how to solve problem faster than other. Teachers naturally come to be regarded as their mission coaching their student to solve as difficult problem as they can.

As mathematical problem can be “solved” by anyone who are informed of the “solution”, all the efforts made by teachers and students are inclined to be the training of the exact knowledge of mathematical formulas and the technical experiences of solving problem. Actually even the highest standard of problems posed in the entrance examination at universities can be often solved only with technical knowledge of how to solve them. In other words, rigorous understanding of basic concepts is seldom needed merely in solving problems. This is a kind of irony for mathematics. The more hours students spend in studying mathematics, the less understanding of mathematics concepts they acquire. But this is just a beginning of vicious circle or negative spiral of mathematics education.

Young students who are enrolled in mathematics department dreaming to be high school teachers suddenly meet the huge gap of studying mathematics and in a lot of Japanese cases they graduate universities without the minimum mastery even of introductory parts of modern mathematics, this is to say, with the same level and the same volume of knowledge they acquire in their high school days.

To find an escape point is the most important task to be done as soon as possible although it is far from trivial.

Moving Forwards by Walking Backwards: Inverse Problems in Te Ara Mokoroa

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We often speak of learning as movement in a forward direction: we progress up a level, we journey along a trajectory. By contrast, the images we use to describe particularly deep kinds of knowledge often imply movement in the inverse direction. “*She knows her topic inside out*”, “*He knows it back to front*”. In mathematics, inverse problems are often more difficult than their direct counterparts: subtracting is harder than adding; dividing is harder than multiplying; factorising is harder than expanding. Yet, mastering a mathematical process in the inverse direction often leads to a deeper understanding of the process in the forwards direction, as well as a deeper appreciation of the generality of the process and its underlying mathematical structure.

Such was the case for eight undergraduate students who worked on inverse problems in calculus and ratios. Although the students were all adept at working in the forward direction (differentiating, or combining two ratios to find a third), they struggled when asked to solve the problem in the inverse direction (antidifferentiating, or finding the ratios that combined to yield a given ratio). The students adapted their knowledge of the problems in the forward direction to make the inverse problem easier, but did so in different ways that revealed the depth of their mathematical understanding in these topics.

Competing or Complementary? Student Perceptions of Live and Recorded Lectures as Learning Resources in Undergraduate Mathematics.

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The use of tablet PC's to deliver and dynamically record lectures at The University of Auckland have been previously documented. Yoon and Sneddon have also reported separately on students' perceptions of effective use of recorded lectures in response to an online survey. This presentation will examine both the methodological process and the outcomes of a subsequent survey given to students in lectures for two undergraduate courses. This short survey asked students if they planned to watch the recorded lecture for that day, and their reasons for attending the lecture in person. The results shed further light on how students view the relative merits of lectures and recorded lectures, and suggest that in the face of increasing availability of resources, students may be developing new learning strategies. A brief description of a new conceptual model being developed by Yoon, which takes an evolutionary perspective on recorded lectures based on these findings, will also be introduced.



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